LIGHT

ARITHMETIC

FOR THE

USE OF SCHOOLS AND COLLEGES

BY

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PREFACE

THIS book is intended for the use of students in Advanced Science Classes. It has been written with the object of giving an account of the chief points of the theory of Optics, both Geometrical and Physical, without making great demands from students in the way of Mathematical attainments. I have made some very elementary applications of the Differential Calculus; but have, in general, given as well alternative methods, or have indicated how the results could be otherwise obtained; and have, I hope, by occasionally presenting the two methods of solution side by side, done something to encourage students in the use of the shorter and more elegant one. The entire book may thus be read without any more advanced knowledge of Mathematics than Higher Trigonometry.

I have throughout kept in view the experimental side of the subject, and have given some account of the methods of making experiments and optical measurements, as well as indicating experiments that can be performed sometimes by very simple means in illustration of important phenomena.

The majority of the illustrations have been engraved from my drawings. For others I am indebted to various books, and in particular to Glazebrook's "Physical Optics" and Jamin's "Cours de Physique." I have also to thank Messrs. Nalder Bros. and Co. for the illustration of their optical bench.

W. T. A. EMTAGE.

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LIGHT

CHAPTER I.

RECTILINEAR PROPAGATION OF LIGHT.

For an object to be seen, there must be light by which to see it. Light is an action taking place in the space between the object seen and the eye which sees it; and which must also, as a rule, exist independently of the object. For most bodies are seen only by means of the modifications which they produce in light; only those which are self-luminous—that is, which could be seen if taken into a dark room—are themselves the sources of the light by which they are seen. As a rule, a body is seen through modifying in various ways light coming from an independent source, as, for instance, the sun or a candle; and, by means of the light so modified and then reaching our eyes, we are able to form a judgment of the body with regard to form, colour, size, and distance. A large part of the study of light consists in considering the various ways in which it is modified by different bodies, either in passing off from the body when it has fallen on its surface, or in passing into it if the body is transparent—that is, if it allows light to pass through it and come out again as light.

Whatever be the nature of the action which we call light, there are certain simple geometrical laws which this action obeys, and the study of which will not depend on the physical nature of the action. The study of these laws will lead to many important results, as, for instance, those which relate to optical instruments; and will explain many optical phenomena, such as the chief phenomena of the rainbow. This part of the study is called **Geometrical Optics**.

The study of the phenomena which indicate the physical nature of light is called **Physical Optics**.

In the geometrical study of light the first thing that we

must notice is its passage in straight lines. In any homogeneous medium, as in a vacuum or in homogeneous atmospheric air, the action by which a small body or single point, as we may call it, of an object is seen at a given point takes place in the straight line joining the body to the point. We may show this by means of the following experiment :—

Take three screens, with small holes pierced in them. Adjust the three holes in a straight line by any means, such as by getting a stretched thread to pass through the three without being constrained by the edges of the middle hole. Light may now be seen to pass through the three holes; but if any one of the screens is displaced, the light is cut off. Again, if the screens are arranged as diaphragms in the same tube, and so adjusted that light passes through the holes, then, however the tube be rotated about its axis, light will continue to pass through. But obviously, if the light has a given path in space between the extreme holes, the middle one can only be so arranged as to suit the passage of the light (however the tube be rotated) if the path is a straight line.

[So well recognized is it that light travels in straight lines, that the readiest method of getting three such apertures in a straight line would be by so arranging them that light from any source passes through all three.]

A **ray** of light, in geometrical optics, must be taken to mean simply a straight line along which light passes; that is, any straight line drawn from the source of light, or from any visible point, is a ray.

A **pencil** is the light in an assemblage of rays in general passing through a single point; that is, it consists geometrically of all the rays contained in a cone, generally of very small angle, having this point as its vertex. An eye sees a visible point by means of all the light in a pencil having that point as vertex, and which enters the pupil of the eye. A pencil may consist, however, of rays coming from or going to a point; that is, it may be either **divergent** or **convergent**. Or, again, the rays may be parallel to each other; then the pencil is called a **parallel** pencil. The vertex of the cone, or point through which all the rays pass, is called the **focus** of the pencil. But we shall have to consider pencils in which there is no single point through which all the rays pass.

A **beam** is the light in any assemblage of rays coming not necessarily from a single point; as, for instance, the light which would come from the whole body of the sun through an aperture formed in a screen.

The words **ray** and **pencil**, in geometrical reasoning and constructions, need not be taken to have any physical significance beyond showing the directions of passage of the light. The straight line joining two apertures, which may be regarded as points, in two screens, is the line along which passes the light that is received by an eye that looks through the apertures simultaneously. This light itself is sometimes spoken of as a ray. But according to the definitions here given it would be a small beam.

Suppose a ray of light to be bent out of its course by any cause. Then by the **deviation** of the ray is meant the angle between its initial and final directions, both directions being reckoned in the sense in which the light travels. So that if, for instance, a ray is bent right back on itself, as by **reflexion**, its deviation is 180°.

Although a ray, as already said, is generally taken to be a straight line, it will frequently be convenient to speak of the path of the light all along its course, even though deviated, as a ray.

Suppose we have a point source of light illuminating a surface, say a screen. Now let an opaque object be interposed between the source and the screen, so as to cut off the light from a part only of the screen. The area from which the light is thus cut off by the object is called the shadow of the object. If we describe a cone, with its vertex at the point and its sides all touching the opaque object, this is called a **shadow-cone**, and gives, by its intersection with any surface, the shadow which the object will cast on that surface. In the case of a theoretical point source, as here considered, the edge of the shadow would be absolutely defined, all points within it being completely cut off from illumination by the source, and all points outside it being uninfluenced by the presence of the shadow-casting object. Such a theoretical shadow is called a **geometrical shadow**. As a matter of fact, any source of light is of finite extent, so that the edge of a shadow cannot be so well defined. And further, on account of the peculiar nature of light, the illumination on the screen must vary continuously, although it may vary very rapidly; that is, it is impossible to pass with absolute abruptness from points in complete shadow to points where the illumination is full. At the same time, the effect due to this cause is so small that very special means must be taken to show it, and we may, as a rule, consider the definition of the edge of the shadow as independent of this cause. Although we cannot

have a point source in practice, simple experiments will show that the more limited the source of light is, the better defined is the shadow. A very small gas flame will give a better shadow than a large one, although the large one gives much more illumination. The sharp shadows, too, given by an arc lamp may be referred to.

Now let us consider an illuminating source of finite extent, such as S, and let a shadow-casting object, O, be placed between this and a screen. By drawing the common tangent lines to

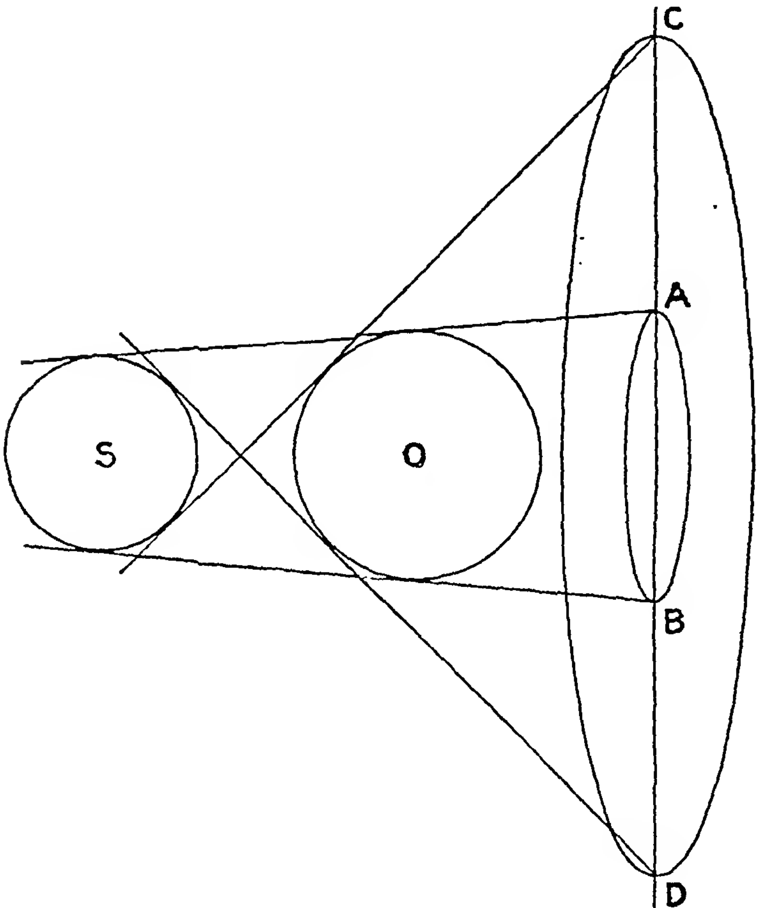


FIG. 1.

S and O, and producing them to meet the screen, we see that they divide the screen into three parts: (1) a region within the curve A B, which is illuminated by no part of S, or such as that S can be seen from no point of it; (2) a region outside the curve C D which is fully illuminated, or with the illumination of which O in no way interferes, or such that S is

fully visible from any point of it; (3) a region between A B and C D, any point of which is only partly illuminated, O cutting off the illumination that would reach this point from a portion of S, or this region is such that S is only partly visible from any point of it. This third region is called a **penumbra.** It is clear that the illumination in it will increase uniformly as we pass outward from the boundary A B to the boundary C D.

A penumbra may exist without a shadow, as the accompanying figure shows. In this case O is too small to cast a

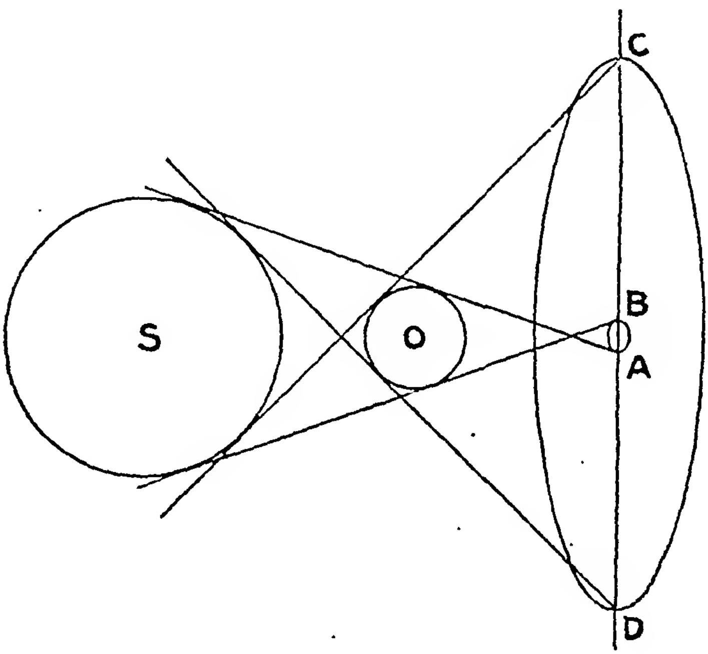


FIG. 2.

complete shadow. The region A B receives illumination from S on both sides of O. The screen could, however, be put nearer to O in such a position as to show a shadow.

Camera Obscura.—If a small hole is made in the shutter of a darkened room, and a screen placed so that light can fall on it through the hole, the objects outside being well lighted, an inverted image of them will be formed on the screen. For from each point without, a small pencil of light will pass through the aperture and illuminate a very small portion of the screen. The assemblage of images so formed of the outside points will be an image of the objects. This will not be very clear and well defined, because to each point there will correspond on the screen a small illuminated area of

size depending on the size of the aperture; and these areas will overlap each other, and thus produce a blurred image. The image may be made sharper by diminishing the size of the aperture; but then the amount of illumination on the screen is diminished.

A similar experiment may be made by holding a sheet of paper with a pin-hole in it between a lighted candle and a screen, when an inverted image of the candle will be produced on the screen.

Law of Inverse Squares.—Suppose we have a source of light, S, whose dimensions are so small compared with the other distances in question that it may be regarded as a theoretical point source. Let this illuminate a surface, A B C D, so that A B C D receives all the light coming from S in the cone S A B C D. Let now A B C D be removed and another surface, A′ B′ C′ D′, be illuminated instead of it. Let this be held parallel to the old position of A B C D, and just twice as far off from S. Let it be of such dimensions as just to

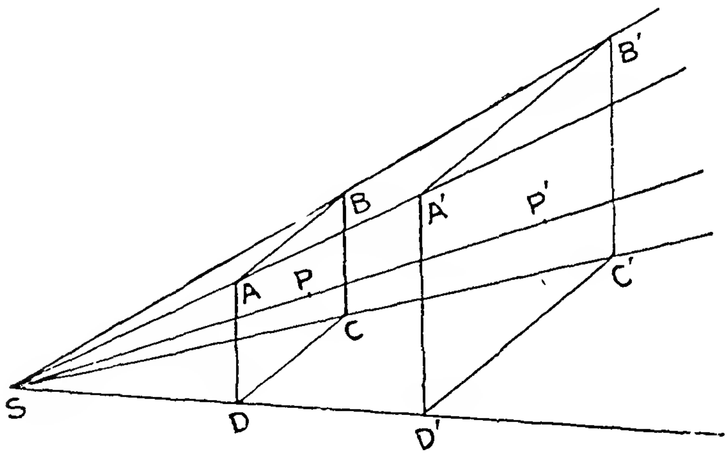


FIG. 3.

receive the illumination which fell on A B C D. It is clear that its linear dimensions must be double those of A B C D. Thus the mean illumination falling on A′ B′ C′ D′ per unit area is one-fourth of that which fell on A B C D. Or again, the mean illuminations per unit area at two points, P, P′, of the two surfaces in the same straight line with S are in the ratio of 4 to 1.

This reasoning may be easily generalized, and we see that, whatever be the distances of S from two surfaces, or two parts of the same surface, *the illuminations per unit area at two points*

where the surfaces are equally inclined to their distances from S, *are inversely proportional to the squares of those distances.*

This law of inverse squares may be verified experimentally. It is comparatively easy to judge whether two contiguous portions of the same surface, such as a screen of white, are equally illuminated, when they are illuminated from sources of the same quality, that is, giving light of the same colour, care being taken that the portion which is to be examined, that is illuminated by either source, is entirely screened from the other. This may be done by placing a rod so that the shadows formed of it on the screen by the two sources are contiguous; or by other methods shortly to be described. The portion in the shadow of the rod formed by one source is then illuminated only by the other source, and we have two adjoining portions of the screen illuminated, one by one source, and the other by the other. Now let five candles of the same sort be taken, and let one be used as one source, and four placed close together as the other source. It will be found that, to make the illuminations equal, the second source must be just twice as far off as the first. Thus we infer that the illumination on the screen due to a candle at a given distance is four times as great as if the candle is at twice that distance. The experiment may be varied by using different numbers of candles.

Law of Cosines.—We have hitherto considered how the illumination received at a point of a surface depends on its distance from the source when its inclination to that distance is unaltered. We shall now consider the effect of that inclination.

Consider a very narrow pencil from a distant point source falling on a surface, the pencil being so narrow that its

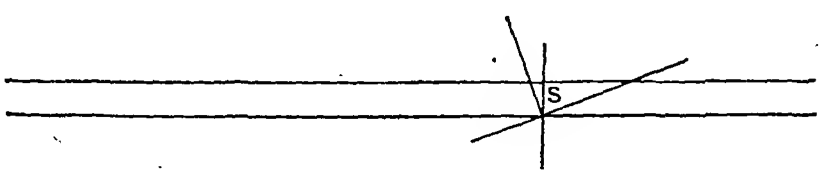


FIG. 4.

rays may be considered parallel. Let the surface at first be normal to these, and let s be the infinitesimal portion of the area illuminated by the pencil. Now turn the surface so that it intersects the pencil at the same distance from the source, but so that its normal where the pencil meets it makes an angle, i, with the rays of the pencil. The area which now

receives the illumination of the pencil is $\frac{s}{\cos i}$; that is, the same amount of illumination is now received by a surface equal in area to the original one $\div \cos i$; or, the illumination received per unit area is equal to what it was in the former case $\times \cos i$. Thus we have the law that the illumination per unit area received by a surface is proportional to the cosine of the angle at which its normal is inclined to its distance from the source of light.

Radiating Surface.—Consider a surface radiating light, and which is equally bright all over, that is, emitting the same amount of illumination per unit area of surface at all points, such as a red-hot ball of metal; or, roughly speaking, the sun would be an example. Experiment shows that to a distant eye all parts of this surface appear equally bright. This means that two portions of the surface subtending equal solid angles, or having the same apparent size at the eye, send equal amounts of illumination to it, no matter what their inclinations may be.

Now consider two small portions of the radiating surface at A and C sending illumination to a distant eye along A B and C D. Let the normal at s be along A B, and that at s' inclined at an angle i to C D. Let s and s' be of such sizes as to appear equally large to the eye. Then (as stated above) experiment shows that s and s' send equal amounts of

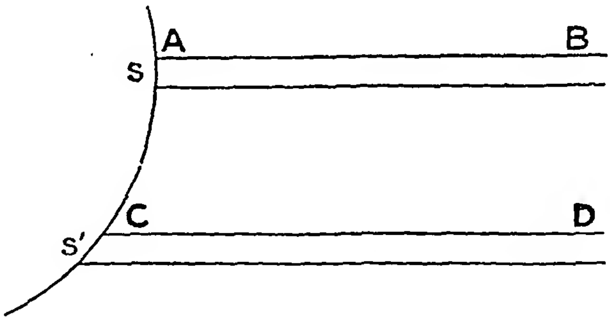


FIG. 5.

illumination to the eye. But the apparent size of s' is the orthogonal section of the cone C D at C, that is, $s' \cos i$. Thus s' emits in the direction C D the same amount of illumination as would be emitted normally by a portion of area $s' \cos i$. And it follows that *the illumination emitted in any direction by a given element of illuminating surface is proportional to the cosine of the angle that the normal to that element makes with that direction.*

Solid Angle.—It will be well to explain here fully what is meant by a solid angle. Suppose we have a sphere of radius unity. The solid angle which any portion of its area subtends at the centre is measured by the number of units of area in that portion. This is also called the solid angle of the cone which has its vertex at the centre and whose sides pass through the contour of the given portion of surface. Now imagine another sphere concentric with the first, and of radius r. Consider the portion of its surface cut out by the same cone. This subtends the same solid angle at the centre as the portion considered of the first sphere. And the portion of its area subtending this angle is $r^2 \times$ the corresponding portion of the other sphere. The solid angle may be measured by *area of sphere* $\div r^2$.

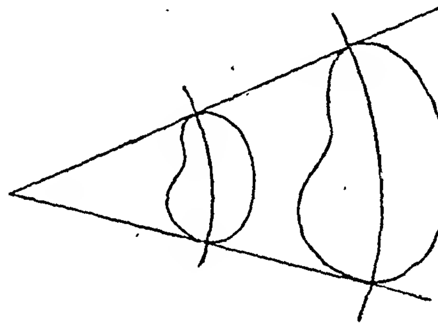


FIG. 6.

Any element of area s subtends at a given point O, distant r from s—r being normal to s—a solid angle whose measure is $\frac{s}{r^2}$. If the normal to s makes an angle, i, with r, to find the solid angle subtended by s at O, we must consider what area at the same distance r from O, and normal to r, would subtend the same solid angle at O. This is the orthogonal

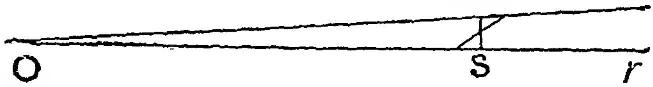


FIG. 7.

section at s of the elementary cone, having O as vertex and s for an oblique section. And the area of this orthogonal section is $s \cos i$. Thus, generally, the solid angle subtended at O by s, at a distance r from O, and having its normal inclined at an angle i to r, is $\frac{s \cos i}{r^2}$.

From the above two laws, namely, the law of inverse squares and the cosine law, it follows that the amount of illumination received at a point from any portion of a uniformly bright surface depends simply on the solid angle subtended by that portion at the point. For suppose an element of area s, at distance r from a given point, and having its normal inclined

at an angle i to r. The illumination which s sends to the point is proportional to $\frac{s \cos i}{r^2}$; but this is the solid angle subtended by s at the point.

Photometry.

It is a matter of considerable practical importance to be able to measure the amount of illumination emitted by a given source or to compare it with that emitted by a standard illuminator. The standard illuminator in general use is a sperm candle burning at the rate of 120 grains per hour. Other more convenient sources may be used, as, for instance, a gas flame with an aperture of definite size through which its light is emitted. Illuminators, it should be noticed, too, are not, as a rule, equally bright all round; it may be necessary to compare the light emitted by them in various directions with the standard light.

By the **illuminating power** or **intensity** of a source of light, we mean the ratio of the light emitted by it to that emitted by the standard source; or the ratio of the illuminations thrown by the two sources normally on surfaces of equal extent and at equal distances from the sources.

By the **intrinsic luminosity** of a surface, we mean the quantity of light, as compared with that emitted by a standard source, emitted per unit area of the surface normally, or emitted per unit of apparent area of the surface in any direction, since, as we have seen, the surface appears equally bright when viewed at any inclination.

To compare the intensities of two sources, the plan always adopted is to cause the sources to illuminate two surfaces, so arranged that they can readily be compared with each other, and to adjust the distances of the sources from the surfaces till the surfaces are equally illuminated, or appear equally bright—care being taken that each surface receives no other illumination than that coming from one of the sources. The distances d, d' of the surfaces from the sources are then measured; and if I, I′ are the intensities of the sources, since the surfaces are equally illuminated, or there is the same amount of illumination per unit area at each of them, we have—

$$\frac{I}{d^2} = \frac{I'}{d'^2}$$

Or—

$$\frac{I}{I'} = \frac{d^2}{d'^2}$$

Thus the illuminating powers are in the direct ratio of the squares of the distances. This operation is called *photometry*, and any apparatus by which it is carried out is a *photometer*. Various photometers have been employed for comparing intensities.

Rumford's, or Shadow Photometer.—The two lights to be compared are placed before a screen, say of white paper, and a rod is placed between them and the screen so as to cast two shadows on the screen. The lights and rod are adjusted so as to make the shadows contiguous and equally dark. The lights and the rod should be nearly *in the same normal to the screen*, so that the portions of surface examined should be illuminated normally. Now, the surface in the shadow formed by each light is illuminated only by the other; so that the illuminations produced by the two lights on the part of the screen examined are equal. If, then, the distances of the lights from the common boundary of the shadows where the illuminations appear equal are measured, the squares of these distances are in the ratio of the intensities.

Bouguer's Photometer.—The illuminated screen is made of ground glass, tissue paper, or some substance which will show the illumination behind. A blackened screen or diaphragm is placed *normally* to this, dividing it into two portions. The two lights are then adjusted close to the blackened diaphragm, so as to illuminate the two portions of the screen into which it is divided by the edge of the diaphragm, equally at their common boundary. By measuring the distances of the lights from this boundary, we can compare the intensities as before.

Bunsen's Photometer.—Bunsen adopted the plan of using a screen in which one portion, in the centre, is made more translucent than the rest by means of a spot of grease. The lights are placed on the two sides of the screen. Then, if the two sides are unequally illuminated, the grease spot will appear darker than the rest on the more illuminated side, and brighter on the other side. By arranging the lights so as to produce similar appearances on the two sides, we know that we have produced equal illuminations.

In Letheby's modification there are two mirrors, M, M', on the two sides of the screen S, so that the two sides may be

seen at the same time by the two eyes. The lights are at I, I′; and the screen with its mirrors can be moved to and fro along a divided scale between them.

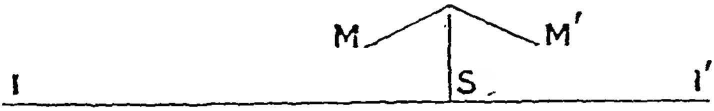


FIG. 8.

Of these photometers the last described gives the best results for purposes of accuracy. Rumford's is a very convenient arrangement for forming a rough estimate of the comparative values of two lights, as the presence of other lights, or of stray light in the room, produces little effect.

In comparing the values of different lights, the question of quality is of great importance, that is, the colours of the lights, one light having more of blue, while another has more of red in it. Two observers may form very different estimates of the comparative values of two lights, simply because of the different ways in which their eyes are affected by them, some eyes being more sensitive to red light and others to blue. Thus the light from an arc lamp is much bluer than that from a gas flame; and so an observer having eyes very sensitive to blue light may give a higher value of the intensity of the arc lamp, when compared with a gas flame, than another would. To get over this difficulty, the following plan is frequently adopted: Two sets of observations are taken, letting the light from the two sources pass in the two cases through red glass and through green glass; and the mean of the results so obtained is taken.

EXAMPLES.

1. A luminous circular disc 1 foot in diameter is placed at 12 feet distance from a screen and parallel to it; an opaque disc 2 feet in diameter is placed symmetrically between them, and at a distance of 4 feet from the screen: find the diameter of the umbra and of the penumbra.

2. A luminous circular disc 10 cms. in diameter is placed at 100 cms. distance from a screen, and parallel to it: find the least distance from the disc at which an opaque square 5 cms. in the side must be placed, parallel to it and the screen, so that there shall be no complete shadow.

3. A uniformly bright surface is looked at through an aperture held close to the eye, and so narrow that the border of the surface is never seen: explain why the amount of light received by the eye is independent of the distance of the surface.

4. In using Letheby's photometer to measure the candle-power of a lamp, the standard light of 2 candle-power is set at a point which is found to be 2·7 cms. on the negative side of the zero of the scale; and the lamp

is set at the reading 130 cms.; the screen receives equal illuminations when it is at 25·5 cms.: find the candle-power of the lamp.

5. Of two small, equally bright surfaces, one is a square and the other a circle whose radius is equal to the side of the square; they are set at distances in the ratio 1 : 2 from a point at which they are to produce equal illuminations: if the circle is at right angles to the line joining it with the point, find how the square must be placed.

CHAPTER II.

REFLEXION. MIRRORS.

A BEAM of light falling on a polished surface undergoes reflexion; that is, after striking the surface, it passes away from it in a definite direction, depending on its direction before meeting the surface. The polished reflecting surface is generally spoken of in optics as a **mirror.**

The manner in which the direction of light falling on a polished surface is modified is specified in the laws of reflexion, which we shall shortly state. These laws refer to a single ray of light falling on the mirror, and reflected off as a ray. They may be taken as experimentally proved for a very limited beam or pencil which approximates to a straight line. And, with regard to their application, for any finite pencil, by tracing the positions of the various reflected rays, we can determine the whole reflected pencil. Before stating the laws, it will be useful to explain the meanings of some terms.

The light falling on a reflecting surface is called *incident* light; and any ray of this light is called an **incident ray.** Any ray of the reflected light is called a **reflected ray.**

Suppose the normal to the mirror at the point where the incident strikes it and where the reflected ray leaves it to be drawn; then the plane containing the normal and the incident ray is called **the plane of incidence,** and the plane containing the normal and the reflected ray is called **the plane of reflexion.**

The angle between the incident ray and the normal is called the **angle of incidence;** the angle between the normal and the reflected ray is called the **angle of reflexion.**

The **Laws of Reflexion** are these—

I. *The incident ray, the normal ray, and the reflected ray are in one plane.*

II. *The incident and the reflected rays make equal angles with the normal.*

These laws may be approximately verified as follows: A circle is divided into degrees. Two tubes are set on it, each having a diaphragm with a small aperture at each end, and with the line of the apertures pointing to the centre of the circle and parallel to its plane, the tube being capable of rotating about an axis through the centre of the circle and at right angles to it. The lines of apertures of the two tubes move in the same plane parallel to the circle. A small mirror is set on the circle, at its centre, at right angles to the line joining the divisions 0° and 180°. Now, if light is allowed to fall through one tube on the mirror, the other tube may be set to receive the reflected light,—which proves the first law; and when it is so set it will always be found to make the same angle with the line 0°–180°, which is normal to the mirror, as the first,—which is the second law.

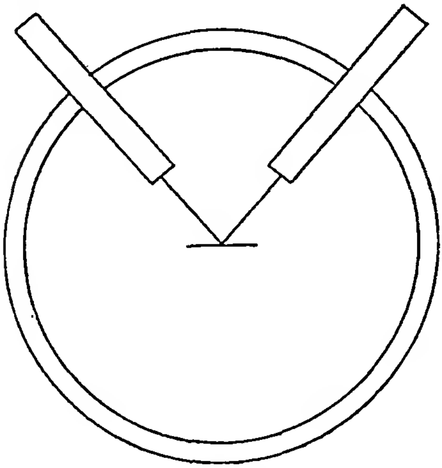


FIG. 9.

This, as has been said, is only an approximate method of verifying the laws, since the experiments described do not admit of very great accuracy. What must be taken as the most rigorous proof is the consistency of the results got by assuming the laws as the basis of observations susceptible of a high degree of accuracy. Similar remarks apply to a great many, if not to most, physical laws; the best proof of the laws being the uniform consistency of the results got by assuming their truth, the experiments adapted for proving them directly not being capable of great accuracy.

As we have said, the light by which we see an ordinary unpolished surface is light which has first fallen on it and then passes off again. This light is sometimes called *diffused*, or *irregularly reflected*. It is generally only a much smaller portion of the light that falls on the surface than regularly reflected light is. It does not follow the laws of reflexion, but passes off from the surface in all directions. There is no perfect reflector. Any surface will diffuse some of the light that falls on it. It is the diffused light only that renders the body visible; regularly reflected light only showing images of other objects.

Image of Point formed by Plane Mirror.—Let A be a luminous or visible point. Let us consider how it would be seen by means of light reflected from a plane mirror. Take any ray from A to the mirror, and suppose this to lie in the plane of the paper: let it be A P. Let the normal from A to the mirror be also in the plane of the paper. Let X Y denote the intersection of the paper with the mirror, which is thus at

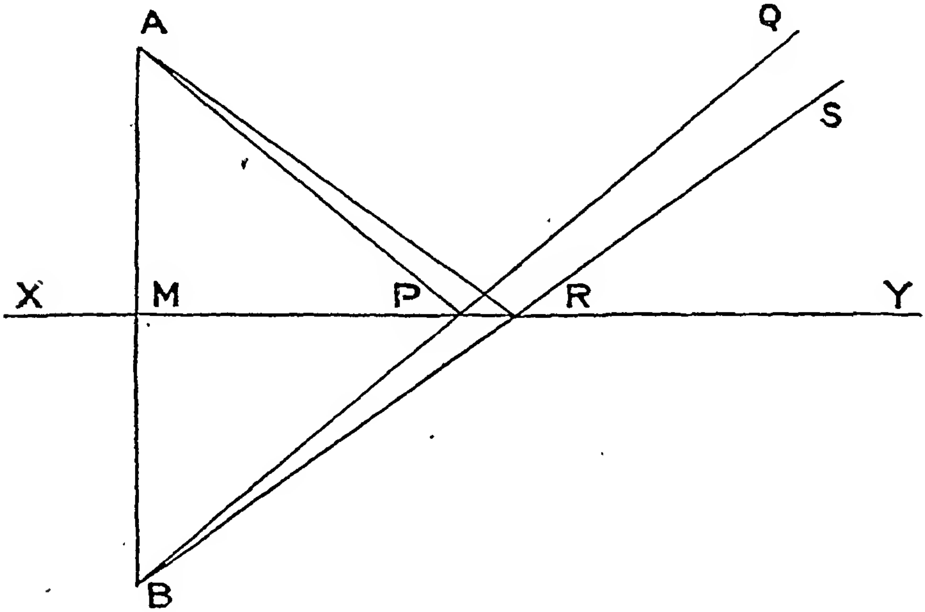


FIG. 10.

right angles to the paper. The plane of the paper, being the plane of incidence, will also be the plane of reflexion. Let P Q be the reflected ray from the point P. A pencil of rays such as P Q, R S, will proceed from the mirror, and would be received by an eye if directed along Q P.

Now draw A M perpendicular to the mirror, and produce it to B, making M B equal to A M. Join B P. Then in the triangles A M P, B M P, the angle B P M = the angle A P M, and ∴ = the angle Q P Y. And Q P is in the same plane with B P and X Y; ∴ B P Q is a straight line. Thus the ray P Q, and similarly all the reflected rays, would, if produced backwards, pass through the point B. To an eye directed along Q P, therefore, the reflected pencil of light which enters it produces the same appearance as a visible point similar to A placed at B, the mirror being removed.

Images.—The point B is called the image of A. It should be noticed that the reflected light is just the same as if it had really come from B. The same action is going on in the

pencil P Q as if A and the mirror were removed and a visible point were placed at B to emit this pencil. An object from whose various points pencils proceed will form an image consisting of the assemblage of images of its various points. If the pencils of light proceeding from the various points of an object undergo change of direction, so that they then proceed, or appear to proceed, from another assemblage of points, this new assemblage of points is called the image of the object. If the deviated rays actually pass through the points of the image, the image is said to be **real**. If the deviated rays only appear to pass through the points of the image, the image is said to be **virtual**. Thus the image of a point formed by reflexion in a plane mirror is virtual.

Image of Object formed by Plane Mirror.—This may be found by constructing the images of the various points of the object. It is such that the straight line joining any point of the object with the corresponding point of the image is bisected at right angles by the reflecting surface. The diagram

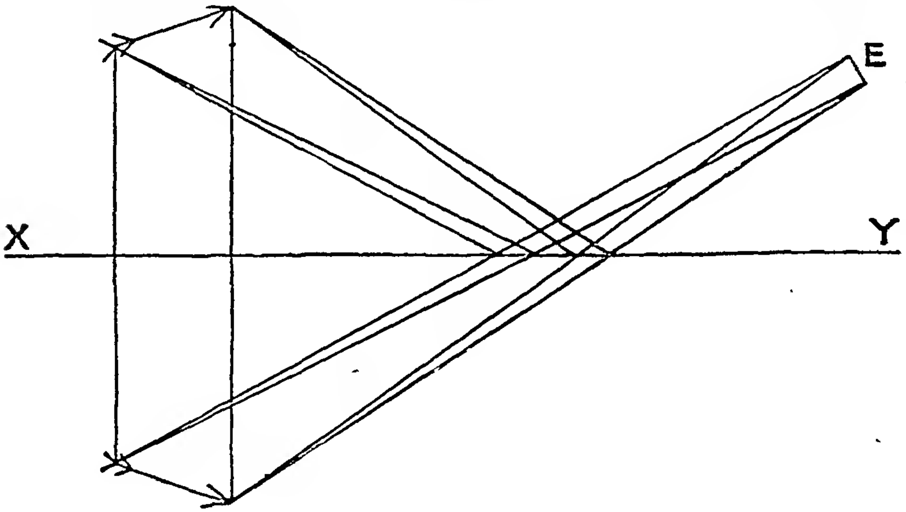


FIG. 11.

shows how the pencils of light from the various points of an object reach an eye situated at E, after reflexion at the mirror X Y, in the same manner as if they had come from the image of the object.

Lateral Inversion.—The points of the image have not in all respects the same relative positions as the points of the object. If, for example, the image formed in a plane mirror of a printed page be observed, then all the letters and words will be inverted, so that, the top and bottom keeping their places, the right-hand side of each letter becomes the left, and

vice versâ, and the words and lines will run from right to left instead of from left to right. If the image of the face be observed, the image of the right side is the left side of the image. This inversion which is produced in the image is called *lateral inversion.*

Reflexion of Convergent Pencils by a Plane Mirror.—Suppose P is the focus of the converging pencils, so that all the rays are, before reaching the mirror, proceeding

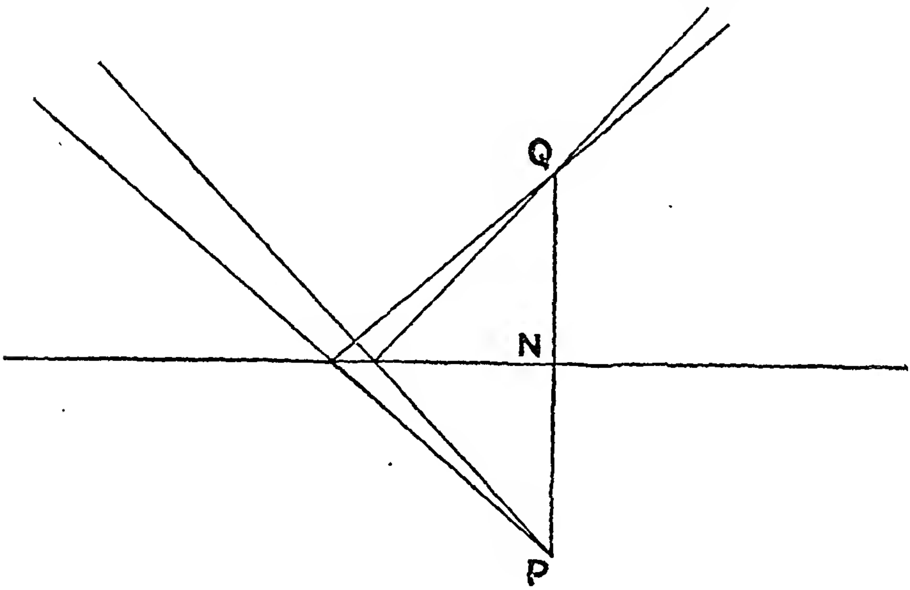


FIG. 12.

to the point P behind the mirror. Draw P N perpendicular to the surface, and take Q on the other side, so that Q N is equal to N P. Then we may show, just as for a diverging pencil, that all the rays after reflexion pass through Q. Thus Q is the focus of the reflected pencil; or Q is the image of P formed by the mirror. An eye in front of the mirror will see a *real* image at Q.

If a real image of an object is produced by any means, and a plane mirror is placed in the way of the pencils, so that the image cannot be formed where it would have been, a real image will be formed by the reflected rays. And this image is constructed from the other, in the same way as a virtual image in a plane mirror from the object, by drawing perpendiculars and producing them.

Reflexion at Several Plane Surfaces.—Pencils of light from a visible object may undergo reflexions at more surfaces than one before reaching the eye. To consider, first, the case of two reflexions; it must be remembered that the light, after

the first reflexion, is proceeding in just the same way as if it had come from an object coinciding with the image formed by the first mirror. To find the result of the second reflexion, therefore, we have merely to consider this image as an object, and so construct the image of it in the second mirror. This, again, may give rise to another image in the first mirror, or perhaps in a third mirror; and so on.

Suppose we have two mirrors only. For an object to be able to give an image in either mirror, it must be in front of the mirror, that is, so that perpendiculars drawn from it to the plane of the mirror meet it on the reflecting side. The same, too, is true for any one of the images. If there is formed in one mirror an image which is behind the other mirror, this image can give rise in the latter mirror to no new image. Rays of light proceeding from it (or, as if they came from it) could not strike the latter mirror.

Two mirrors at right angles.—Call the mirrors M_1, M_2.

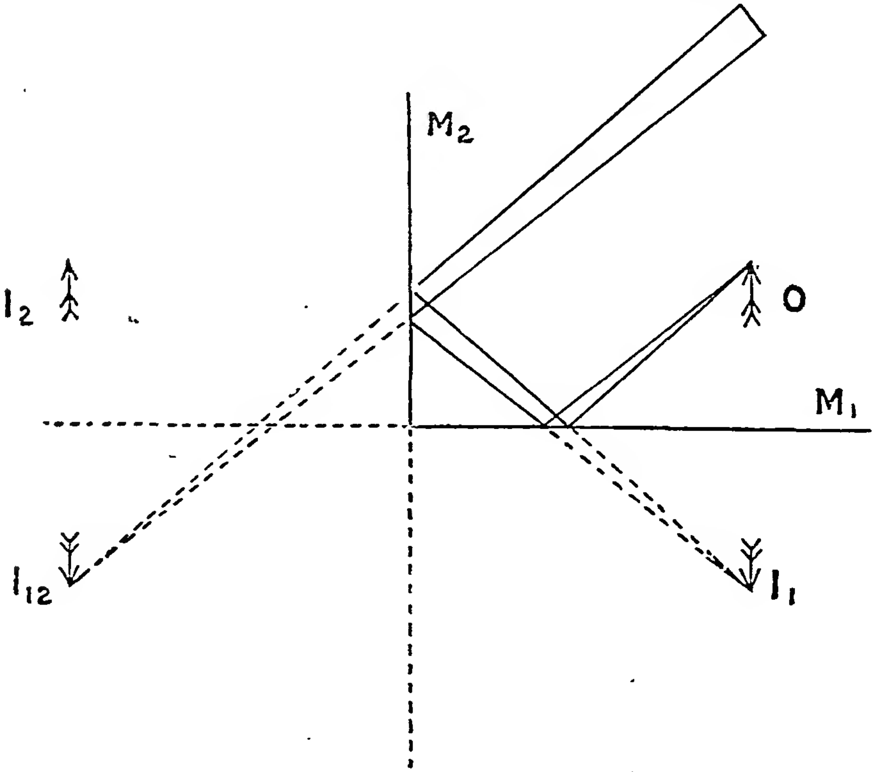


FIG. 13.

Let O be the object. This will give rise to images I_1, I_2 in the two mirrors. I_1 will give in M_2 an image which may be

called I_{12}. This would be seen by an eye so situated that M_2 lies between it and all points of I_{12}. Again, I_2 will give in M_1 an image which we may call I_{21}, but which will coincide exactly with I_{12}, so that they may be called one image, I_{21}; that is to say, an image formed by reflexion first in M_2 and then in M_1 would be seen by an eye so situated that M_1 comes between it and all points of the image I_{21} or I_{12}. An eye may be so situated that the image of O formed by two reflexions is partly I_{12} and partly I_{21}. The intersecting edge of the mirrors would be seen to cross this image, and a part of it would be seen in each; but if the mirrors are properly adjusted, these two parts would fit exactly, wherever O may be. In this case three images are formed. A pencil of light by which I_{12} is seen is shown in the diagram.

Two mirrors inclined at 60°.—The diagram shows the images that are formed. The image marked $I_{\substack{212\\121}}$ may be formed by reflexions in the mirrors either in the order M_2,

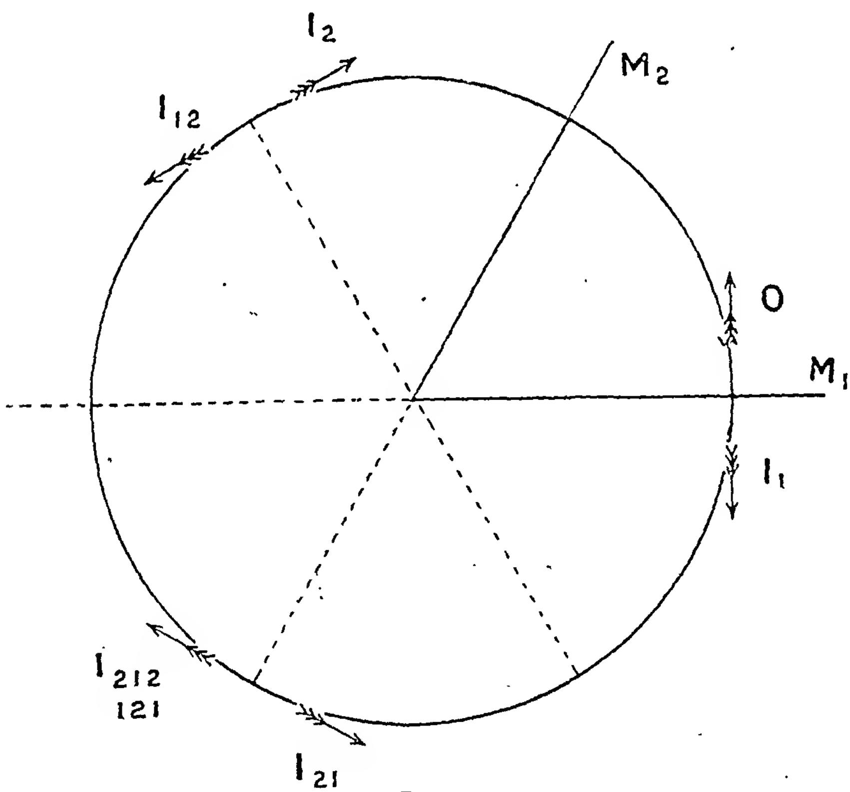


FIG. 14.

M_1, M_2, or in the order M_1, M_2, M_1, according to the position of the eye. In this case five images are formed.

It may be seen that, in general, the two cases considered being particular examples, if two mirrors are inclined at an angle which is the nth part of four right angles, n being an integer, there will be $n - 1$ images.

It should be noticed, as the above diagram indicates, that the object and its images are arranged on the circumference of a circle with its centre at the intersection of the mirrors. The symmetry of the arrangement of the object and the images in pairs should also be noticed.

The Kaleidoscope.—This optical toy was invented by Sir David Brewster. It consists of three, or sometimes of two, equal strips of silvered glass, arranged at angles of 60°, and having their reflecting sides turned towards each other, and contained in a tube. At one end of the tube is a pair of transparent glass plates set across the axis of the tube, and containing pieces of coloured glass or other small objects. On looking through the other end of the tube, there is seen a symmetrical arrangement of similar patterns, consisting of the figure seen through the triangular aperture and its various reflexions. As the pieces in the end are moved about by turning the tube, these patterns continually change. The kaleidoscope is employed by designers.

In all the cases we have considered, the number of images formed is finite. The number may, however, be, theoretically, infinite, although the brightness of the images gradually becomes less and less.

Two parallel mirrors.—In this case there is an infinite

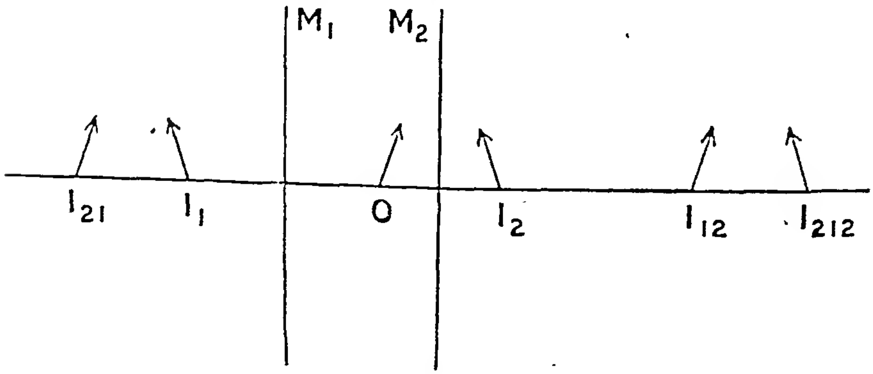


FIG. 15.

succession of images arranged, with the object, along a straight line perpendicular to the mirrors, as the diagram shows.

A pretty experiment may be performed with two mirrors and a lighted candle. Place the candle between the mirrors,

and hold these inclined at a small angle. A series of images of the candle will be seen arranged in a circle in pairs equally distant all the way, or with equal distances between each two successive images if the candle is equidistant from the mirrors. As the mirrors are brought nearer and nearer to parallelism, the circle enlarges, and more images appear; and when the mirrors are parallel, the circle becomes the straight line of the last case.

Measurement of Angular Deflexion by Reflected Light.—Poggendorff's Method.—To measure the angle through which a body is rotated about the vertical, a vertical mirror is attached to it. Let a horizontal ray of light fall on this mirror. Suppose it to make an angle, α, with the normal. Then the reflected ray will make an angle, 2α, with the incident ray. If, now, the mirror is rotated through an angle, β, the reflected ray will make an angle, $2\alpha \pm 2\beta$, with the incident ray; that is, it will be deflected through an angle, 2β. Thus if we measure the deflexion of the reflected light, that of the body to which the mirror is attached is half as much.

The deflexion of the reflected light may be measured in the following manner. The figure shows a plan of the arrange-

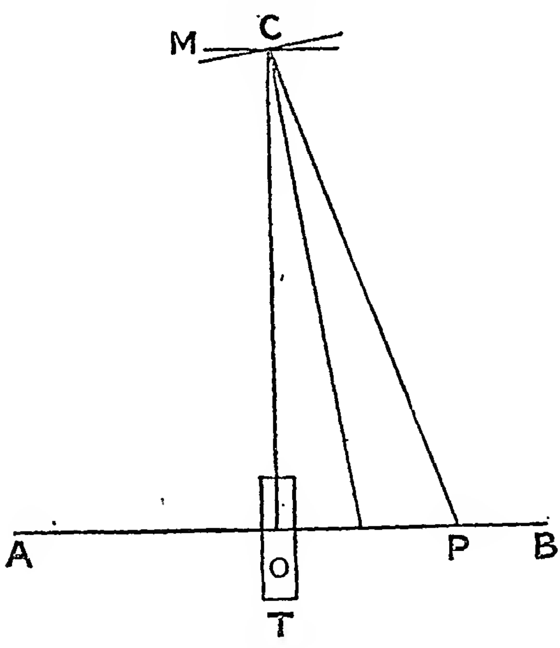


FIG. 16.

ment. A graduated scale, A B, is placed on about the same level as the mirror M, and parallel to it in its undeflected position. Let d be the distance between them. A B is viewed

by reflexion in M by means of a telescope, T, placed just under A B. When M is undeflected, the reading of the scale at O, just above the telescope, should be seen at a marked position in the telescope, generally to coincide with a vertical wire fixed in the telescope. C O is perpendicular to A B. When the mirror is deflected through an angle, θ, suppose the reading at P is seen in the telescope in the marked position. Then we have $\tan 2\theta = \frac{\mathrm{OP}}{d}$.

SPHERICAL MIRRORS.

A spherical mirror is a portion, generally very small, of a reflecting spherical surface. If the reflecting surface is on the concave side, towards the centre of the sphere, the mirror is called **concave**; if on the convex side, away from the centre of the sphere, the mirror is called **convex**. The centre of the sphere of which the mirror is a portion is called the **centre**, or **centre of curvature**, of the mirror. If the mirror has a circular boundary, its middle point is called its **pole**. And the straight line joining the centre to the pole is called the **principal axis**.

We shall investigate the relative positions of a small object and its image formed in a spherical mirror. The result is conveniently expressed in a formula. For the purposes of proving and applying this formula, it is necessary to have a convention with regard to the algebraical signs of the distances —which will all be measured from the pole of the mirror, and along its axis—occurring in the formula. The convention is that all distances measured *towards* the direction *from* which light is coming, or measured in the direction opposite to the incident light, are reckoned *positive*, and those measured in the *opposite direction* are reckoned *negative*. Thus with regard to the sign of the radius of the mirror: for the concave mirror, the light, striking the concave side, is travelling in the direction from the centre to the pole, so that the radius, measured from the pole to the centre, is positive; for the convex mirror the radius is seen, in a similar way to be negative.

Consider the formation of an image of a single visible point in a concave mirror. We shall suppose the point to be situated on what we have called the principal axis of the mirror; but is it clear that, wherever the point may be, if the straight line joining it to the centre meets the mirror, an image of it will be formed in the same manner with reference to this straight line.

Let the figure represent the section of the mirror by the plane of the paper, C being the centre, and A the pole. Let P be a visible point on the axis. Draw P R, a ray from P to the mirror. To find the position of the reflected ray from R, first it must lie in the plane with P R and C R, which is normal to the mirror at R; that is, it must lie in the plane of the paper, or

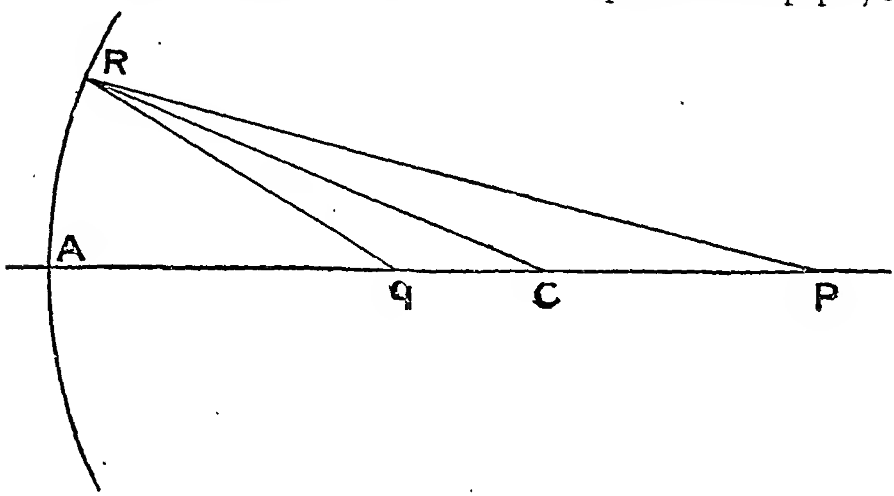


FIG. 17.

must intersect the principal axis (thus we see that all the reflected rays must meet the radius through P); next it must make the angle q R C equal to the angle P R C. Now, all the rays from P do not, after reflexion, pass through the same point on the axis. But there is an ultimate limiting position of the point through which rays will pass if they are taken indefinitely close to the axis. This is the point at which a true image of P will appear to an eye looking along P C A, or looking just a little to the side, so that P is not in the way of the image; this image being formed by rays meeting the mirror indefinitely near to A. An eye would receive light which comes from P by reflexion at any other part of the mirror, but nowhere else would a true and distinct image be seen. This case differs from that of the plane mirror. A true image is seen by reflexion at any part of a plane mirror, and always in the same position. Notice that here a *true* image has been spoken of; this must be distinguished from a *real* image. A *true* image may be *real* or *virtual.*

Let Q be the ultimate limiting position of q, as the rays from P which, after reflexion, pass through q become indefinitely close to the principal axis. Let the radius of the mirror $= r$. Let the distance of object from pole $= u$; and distance of image from pole $= v$. So that u and v in our

figure are both positive as well as r. We wish to find the relation between u, v, and r. From the figure we have, since R C bisects the angle P R q—

$$\frac{\mathrm{RP}}{\mathrm{R}q} = \frac{\mathrm{CP}}{q\mathrm{C}}.$$

In the limit, when R becomes indefinitely near to A, so that q coincides with Q, this relation becomes—

$$\frac{u}{v} = \frac{u-r}{r-v};$$

$$\therefore u(r-v) = v(u-r),$$

$$r(u+v) = 2uv.$$

This may, again, be written, on dividing by $r\,u\,v$—

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}.$$

This relation is quite general for spherical mirrors, concave or convex; and for all positions of object and image, provided due regard is had to the signs of u, v, r. We have taken here the simplest, and what is called the typical, case, in which all three quantities are positive. But if any other case is examined, the same formula will be found to hold. Let us, for example, examine the case of the formation of an image in a convex

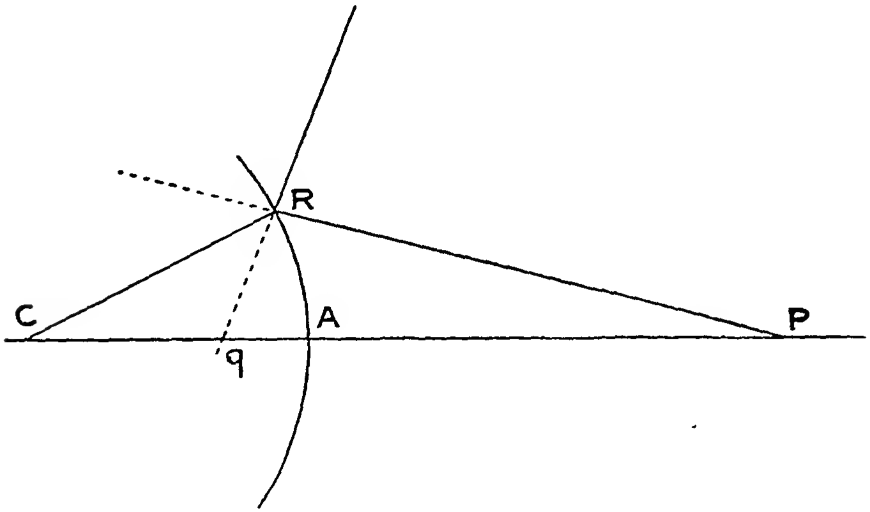


FIG. 18.

mirror. Let the visible point P (Fig. 18) on the principal axis send out rays, of which one, P R, after reflexion, appears to have come from the point q on the axis. The limiting position of q,

when P R is indefinitely close to the axis, is a point, Q, the image of P. In the figure, C R bisects the exterior angle of the triangle P R q, got by producing P R. So that we have—

$$\frac{RP}{Rq} = \frac{CP}{Cq}.$$

In the limit this becomes—

$$\frac{AP}{QA} = \frac{CP}{CQ}.$$

This is a geometrical relation existing among the lengths of the lines, all four of which lengths are, here, to be taken as positive.

Let us now pass to the algebraical relation, and introduce the symbols r, u, v.

A P $= u$, u being positive—

Q A $= v$ in absolute magnitude; but v is here negative, so we must write—

$$\begin{aligned} QA &= -v. \\ CP &= CA + AP = -r + u. \\ CQ &= CA - QA = -r + v. \end{aligned}$$

Thus our relation becomes—

$$\frac{u}{-v} = \frac{-r + u}{-r + v}.$$

Changing the signs of both denominators, this gives, as before—

$$\frac{u}{v} = \frac{u - r}{r - v}.$$

Or—

$$\frac{1}{u} + \frac{1}{v} = \frac{2}{r}.$$

In this formula r may be positive or negative, according as the mirror is concave or convex; v may have either sign, its value depending on r and u. But u would, for such cases as we have considered, always be positive from the nature of the case; that is, whenever it denotes the distance from the mirror to a visible point.

There are, however, cases in which u may be negative; and for these the formula holds too. Such a case is this: Suppose we have a convergent pencil of light, such, for instance, as would be produced by a concave mirror in the way we have examined. Now let a mirror be placed so as

to receive this pencil normally before it converges to a point, and reflect it so that it either converges to a point in front of the mirror after reflexion, or appears to diverge from a point behind the mirror. Then the point to which the pencil was converging takes the place of the object, and may in this case be called a **virtual object.** The other point to which the pencil converges after reflexion, or from which it appears to diverge, is the image of this object. The distance u, measured from the mirror to a point behind it, is negative.

The points P and Q are called **conjugate foci.** In the formula, u and v are interchangeable; they occur in the same way. If u and v have certain values satisfying the formula, then, if we give to u the value of v, v takes the value of u, Therefore if in any case an object at P gives an image at Q, an object at Q would give an image at P.

Principal Focus.—Focal Length.—There is one point on the principal axis of a mirror of great importance. Suppose that P goes off to an infinite distance; or, what is the same thing, suppose a parallel pencil of light, indefinitely small and indefinitely close to the axis, to strike the mirror. Through what point do the rays pass, or appear to pass, after reflexion? The formula shows this. Making $u = \infty$, we get $v = \frac{r}{2}$. That is, the point is, for either mirror, midway between the pole and the centre. This point is called the **principal focus** of the mirror. We shall, in general, denote it by the letter F. The distance of F from the pole is called the **focal length** of the mirror. In both cases it is $\frac{r}{2}$.

For a concave mirror, the principal focus is the point through which the reflected rays from a parallel axial pencil actually pass; or it is the real image of the point at infinity on the principal axis. It is called a **real principal focus.** The *focal length* is *positive.*

For a convex mirror, the principal focus is the point through which the reflected rays from a parallel axial pencil appear to pass; or it is the virtual image of the point at infinity on the principal axis. It is called a **virtual principal focus.** The *focal length* is *negative.*

If in the formula we make $u = \frac{r}{2}$, we get $v = \infty$. From this, or from what has been said about the interchangeability of P and Q, it is seen that the principal focus has also this

property. An axial indefinitely narrow pencil diverging from F for a concave mirror, or converging to F for a convex mirror, will, after reflexion, be a parallel pencil along the axis.

Let us denote the focal length of a mirror by f, so that $f = \frac{r}{2}$; then the formula may be written—

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

To examine the various ways in which a concave or a convex mirror may act on an axial pencil, let us write the formula—

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u}.$$

The figures show the action in the various cases.

1. For a *concave* mirror—

(1) u positive; v may be positive.

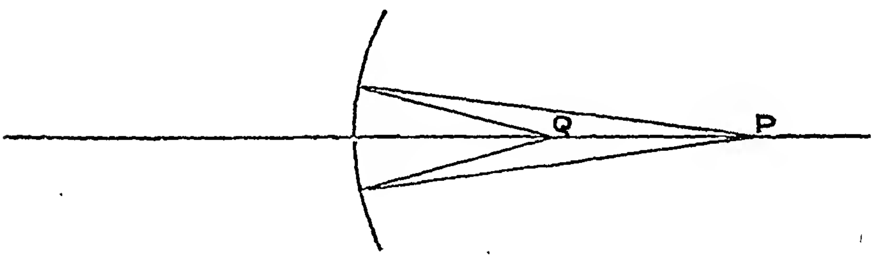


Fig. 19.

(2) u positive; v may be negative; v is then numerically greater than u.

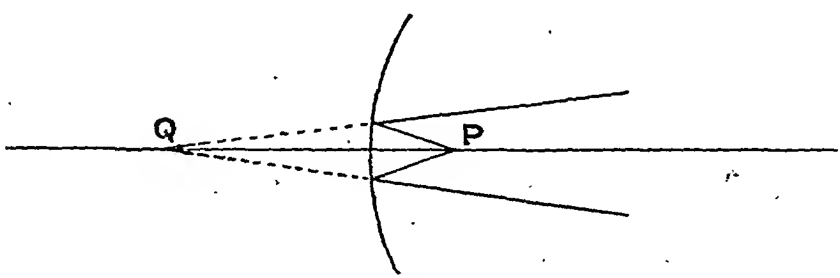


Fig. 20.

(3) u negative; v must be positive (Fig. 21).

In any case, it is seen that the pencil is more convergent (or less divergent) after reflexion. Thus the mirror is called a *converging mirror*.

2. For a *convex* mirror—

(1) u positive; v must be negative (Fig. 22).

(2) u negative; v may be positive; v is then numerically greater than u (Fig. 23).

(3) u negative; v may be negative (Fig. 24).

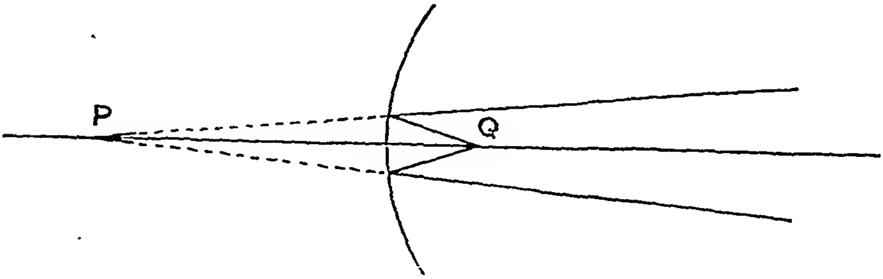


FIG. 21.

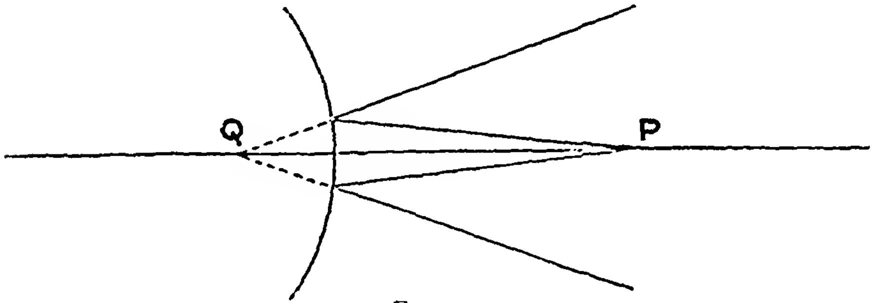


FIG. 22.

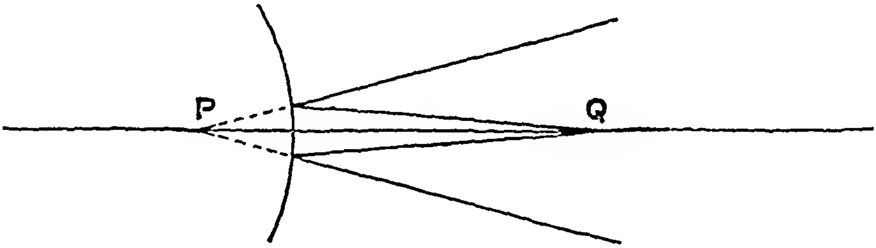


FIG. 23.

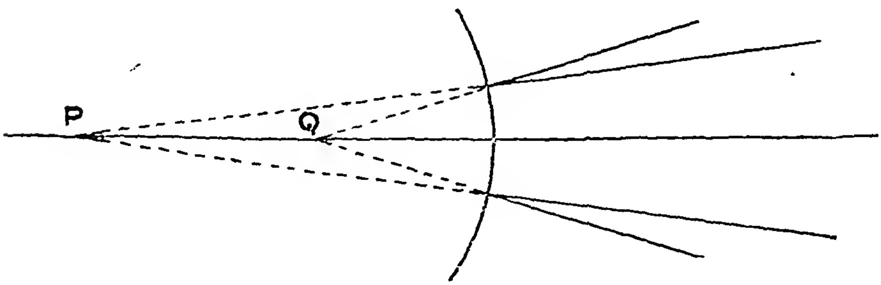


FIG. 24.

In any case, it is seen that the pencil is more divergent (or less convergent) after reflexion. Thus the mirror is called a *diverging mirror*.

We shall now consider the variations in position of the image as the position of the object is changed. We shall make use of the formula—

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}.$$

1. *Concave mirror*—

When $u=\infty$, $v=f$; *i.e.* object at infinity gives image at F.

As u decreases in value from ∞ to f, $\frac{1}{v}$ increases from 0 to $\frac{1}{f}$; $\therefore \frac{1}{v}$ decreases from $\frac{1}{f}$ to 0; $\therefore v$ increases from f to ∞; *i.e.* as the object comes in from infinity to F, the image goes out from F to infinity.

When $u=r$, $v=r$; *i.e.* object and image meet and coincide at the centre.

As u decreases from f to 0, v, which is now negative, increases algebraically from $-\infty$ to 0; *i.e.* as the object moves in from F to the mirror, the image, which is virtual, comes from an infinite distance behind to the mirror.

It should be noticed that the object gives a real or a virtual image according as it is beyond or within the principal focus.

2. *Convex mirror*—

When $u=\infty$, $v=f$; *i.e.* object at infinity gives image at F.

As u decreases from ∞ to 0, v is always negative, and increases algebraically from f to 0; *i.e.* as the object comes in from infinity to the mirror, the image moves up from behind from the principal focus to the mirror.

It should be noticed that the image is always virtual.

We have only considered the cases in which the object is a real visible point; and the general conclusions drawn refer only to such cases. For instance, with a pencil of light converging to a point between the principal focus and pole of a convex mirror, the image formed would be real. Such cases as this may be considered just as the others by the help of the formula.

Image of Small Object.—Let A B be an object of dimensions indefinitely small compared with the radius of the mirror and the distance of the object from the mirror. The image of A B is the assemblage of the images of its various points. Every point of A B is at nearly the same distance from the mirror. This will be very approximately true indeed (or true to small quantities of the second order) if the small object is all in a plane perpendicular to the principal axis. u having the same value for all points, it follows that v will

have the same value, and the image is a small image at *a b*. When A B is all in a plane perpendicular to O C, *a b* is similar

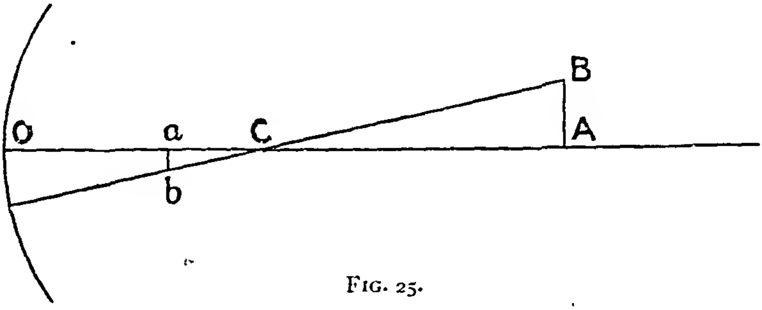


FIG. 25.

to it, being in another plane perpendicular to O C, and every straight line joining corresponding points passing through C.

Graphic Construction of Images.—We shall now consider a convenient method of constructing graphically the image, formed in a mirror, of a small object. These constructions will depend on the following principle: A small pencil near the axis gives by reflexion a pencil with a definite focus. This focus we can find if we can construct geometrically any two different rays of the pencil. For the focus is the point of intersection of all the rays. Now, we shall see that there are three rays of the incident pencil which, after reflexion, we can follow up and draw. These are the following:—

(1) The ray which passes through the centre: this strikes the surface normally, and is reflected through the centre.

(2) The ray which comes in parallel to the axis: this is reflected through the principal focus.

(3) The ray which comes in through the principal focus: this is reflected parallel to the axis.

Of course, when the mirror is convex, and the centre and principal focus are behind it, we must consider rays whose *directions* are through these points.

We shall now consider the three different general cases that may occur of the formation of an image of a small object in a spherical mirror, giving the graphic construction for each case, the object being real.

All the rays drawn in these constructions are supposed to be indefinitely close to the axis. Therefore, to represent them in a figure of finite dimensions, we must suppose this figure to be very greatly spread out in the direction at right angles to the axis, but not in the direction of the axis. The trace of the mirror on the plane of the paper will thus be a straight line at right angles to the axis.

I. *Concave Mirror: Object beyond principal focus.*—Take A B, a small object, in a plane perpendicular to the axis, with the point A on the axis. The rays B R, B S, B T, from the point B of the object, are respectively through the centre, parallel to the axis, and through the focus; and they are reflected through

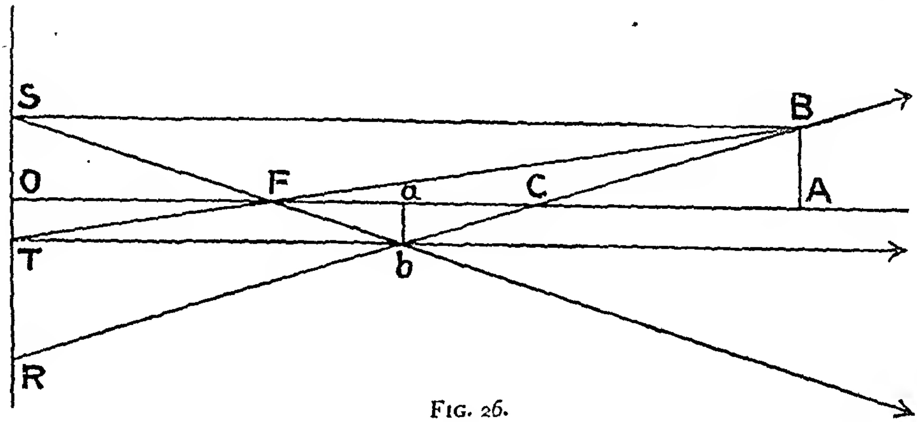


FIG. 26.

the centre, through the focus, and parallel to the axis. They give by their intersection the point *b*, the image of B. *b* is seen by means of a pencil diverging from it, of which there are three rays, these rays being indicated by arrow-heads. The position and magnitude of the whole image *a b* will be found by drawing *b a* perpendicular to the axis.

A B and the corresponding image may be drawn in other positions; for instance, if A B comes between the mirror and C, *a b* will be beyond C. But as long as A B is beyond F, the general characteristics of *a b* remain the same.

The image is *real* and *inverted.*

II. *Concave Mirror: Object between mirror and principal*

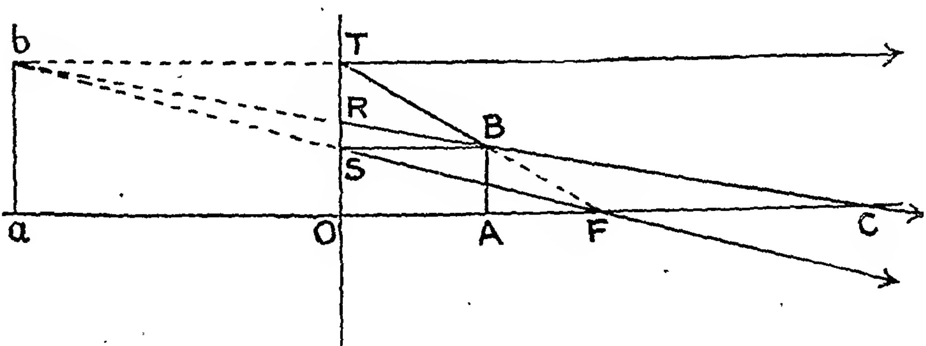


FIG. 27.

focus.—The same three rays as before, and indicated by the same letters, are here drawn. The pencil they give has here a

virtual focus, *b*. The entire image *ab* is constructed as before. Notice that the action on the right-hand side of the points R, S, T is physically the same, and produces the same effect on the eye, as if it came from a real visible point *b*.

The image is *virtual* and *erect*.

III. *Convex Mirror*.—The construction follows the same

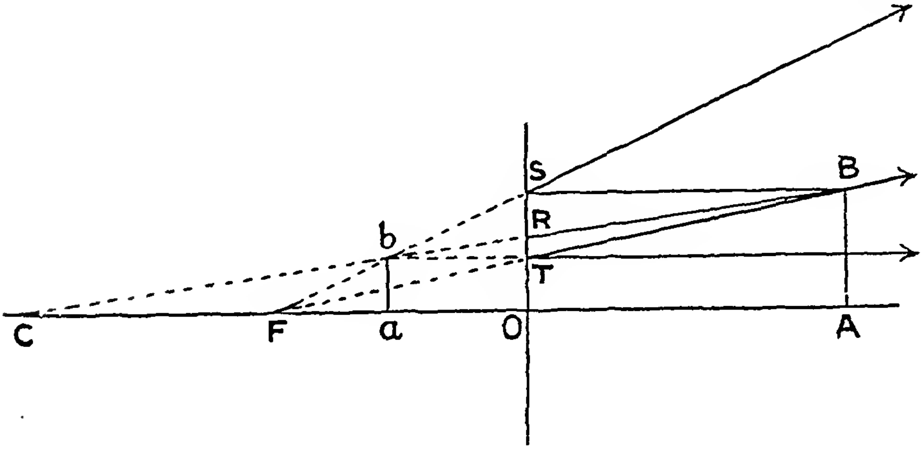


FIG. 28.

lines as before. However A B is moved about in front of the mirror, the image will have the same general characteristics.

The image is *virtual* and *erect*.

In any case the image is necessarily erect if it and the object are on the same side of C, and inverted if they are on opposite sides; for the rays joining corresponding points of object and image meet in C.

The image is real if on the same side of the mirror as the object, for then the rays really pass through the image; it is virtual if on the opposite side.

By referring to all the cases, it will be seen that the image is real when inverted, virtual when erect.

Magnification.—When an image of a small object is formed in a mirror, the ratio of any linear dimension of the image to the corresponding dimension of the object—both being measured at right angles to the axis on which object and image are situated—is called the **magnification** produced by the mirror.

In all the three figures that have just been given, the magnification will be the fraction $\frac{ab}{AB}$. Important expressions for the magnification may be deduced by reference to these

figures. What follows is applicable to any of these figures, and all three of the figures should be referred to at each step of the work. Let us denote the magnification by m. We have—

$$m = \frac{ab}{AB}.$$

From the similar triangles *abc*, ABC, we get—

$$m = \frac{ab}{AB} = \frac{Ca}{CA};$$

$$\textit{i.e. } m = \frac{\text{distance of image from centre}}{\text{distance of object from centre}} \quad . \quad . \quad (1)$$

This expression, which in symbols is $\frac{u-r}{r-v}$, is also equal to $\frac{u}{v}$ (see p. 25); so that we have—

$$m = \frac{\text{distance of image from mirror}}{\text{distance of object from mirror}}. \quad . \quad . \quad (2)$$

Approximately, O S is a straight line at right angles to O C, and equal to A B. Thus from the similar triangles *ab*F, OSF, we get—

$$m = \frac{ab}{AB} = \frac{ab}{OS} = \frac{aF}{OF};$$

$$\textit{i.e. } m = \frac{\text{distance of image from principal focus}}{\text{focal length}} \quad . \quad (3)$$

Approximately, O T is a straight line at right angles to O C, and equal to *a b*. Thus from the similar triangles OTF, ABF, we get—

$$m = \frac{ab}{AB} = \frac{OT}{AB} = \frac{OF}{AF};$$

$$\textit{i.e. } m = \frac{\text{focal length}}{\text{distance of object from principal focus}} \quad . \quad (4)$$

The numerical values, merely, of the quantities expressed in these fractions for m are to be understood. Thus for a convex mirror O F is not the focal length, but is − focal length.

These four values of m may be expressed in symbols as follows: The magnitudes whose ratios we shall write down are all so taken that they are positive in the first figure; but their numerical magnitudes will give the magnification in any case.

$$m = \frac{u - r}{r - v} = \frac{u}{v} = \frac{v - f}{f} = \frac{f}{u - f}.$$

Of the expressions given, perhaps number 4, which involves the position of the object and not that of the image, is the most important.

Some general inferences may be drawn with regard to the magnification produced in the various cases. If the magnification is greater than 1, so that the image, measured as specified, is greater than the object, the image is said to be *magnified;* if the magnification is less than 1, the image is said to be *diminished.*

Concave Mirror.—When the object is beyond the centre, it is at a distance from F greater than f, so that from the value (4) of m the image is *diminished;* when the object is at C, the image is equal to it; when the object is between the mirror and the centre, the image is *magnified.*

Convex Mirror.—The object is always at a greater distance from F than f; so that from the value (4) of m the image is always *diminished.*

We shall now enumerate, for reference, the characteristics of the images formed by either mirror, and for all positions of the object, with regard to the following four particulars :—

(1) Position;
(2) Whether real or virtual;
(3) Whether erect or inverted;
(4) Size.

Concave Mirror.—Object beyond C; image between F and C, real, inverted, diminished.

Object at C; image at C, real, inverted, equal to object.

Object between F and C; image beyond C, real, inverted, magnified.

Object between O and F; image behind mirror, virtual, erect, magnified.

Convex Mirror.—Object in front of mirror; image behind mirror, between F and O, virtual, erect, diminished.

Spherical Aberration.—The rays of light in a broad pencil will not, after reflexion, give a pencil with a definite focus. An indefinitely narrow axial pencil gives, as we have seen, after reflexion, a pencil with a definite focus. But as rays are taken more and more inclined to the axis, they will, after reflexion, meet the axis farther and farther away from this focus. The figure shows how the rays from P, as they are taken at greater inclinations to the axis, meet the axis after reflexion at points farther from Q. This *aberration* of the

reflected rays may be considered as due to the form of the mirror, as it would be possible to construct a mirror which

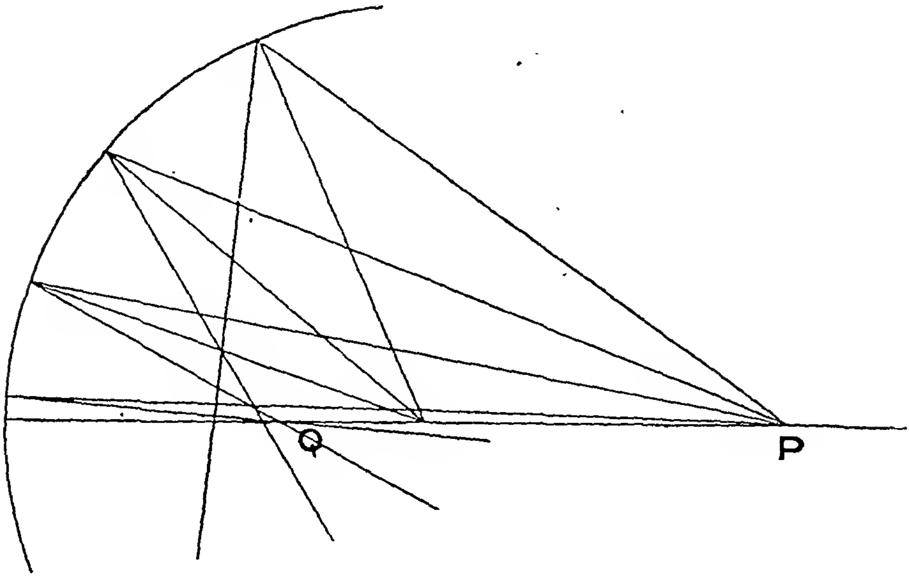


FIG. 29.

would bring all the rays to one focus. It is called **spherical aberration.**

We shall see now that mirrors may be made which, for particular broad pencils, do not produce any aberration.

Parabolic Mirror.—This is of the form of a paraboloid of revolution (got by rotating a parabola about its axis). If F

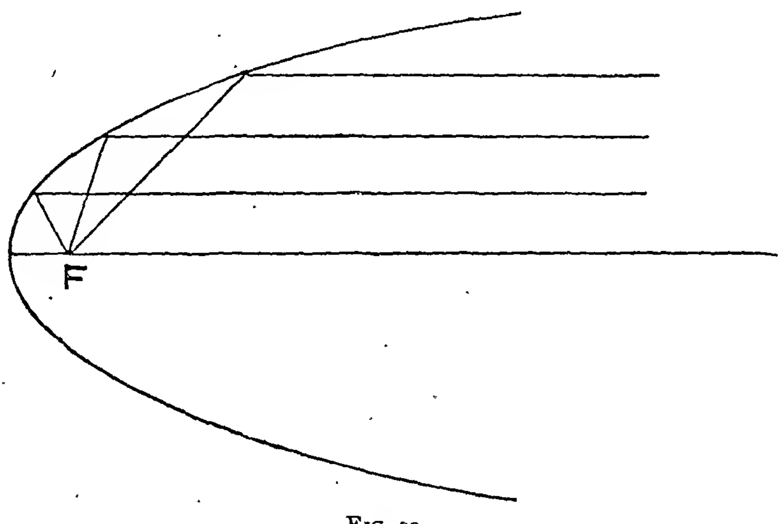


FIG. 30.

is the focus of the paraboloid, that is the principal focus of the mirror; the normal at any point is equally inclined to the axis

and to the straight line joining that point to F. Thus any ray coming in parallel to the axis will be reflected to F. F is the focus for all rays parallel to the axis.

Again, all the rays coming from a luminous or visible point at F would be reflected parallel to the axis. Thus a parabolic mirror is frequently used for throwing a strong beam of light in a definite direction. A lamp is placed at the focus, and the axis of the mirror turned in the required direction.

Ellipsoidal Mirror.—This is of the form of an ellipsoid of revolution (got by rotating an ellipse about its major axis). If F, F′ are the foci of the figure the normal at any point

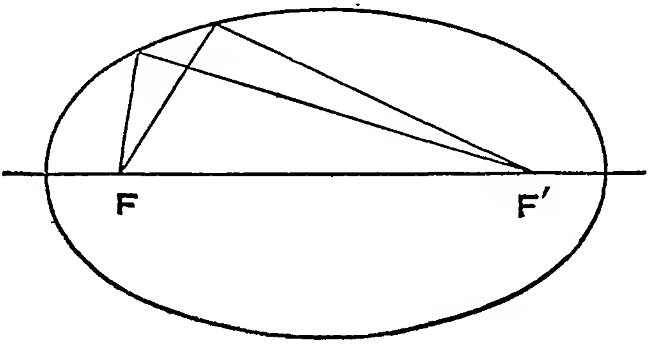


FIG. 31.

makes equal angles with the straight line joining the point to F and F′. Thus any ray from F is reflected to F′. Either of the points F and F′ is the focus for all rays coming from the other.

Determination of Focal Length.—The focal length of a spherical mirror may be found by many methods.

If the radius of curvature is found in any manner, the focal length, which is half of the radius, is known.

By measuring corresponding distances from the mirror of a small object and its image along the line joining them, and substituting in the formula—

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

the value of the focal length of a concave mirror can be found.

If we get a small object into such a position that its image in a concave mirror coincides with itself, then the object and image are both at the centre. Thus we can determine the radius, and so the focal length.

The focal length of a convex mirror may be found by taking a pencil of converging light, and noticing the position

of its focus, then placing the mirror so that the light falls on it normally and is reflected as a convergent pencil; that is, the original focus must be between the mirror and its principal focus. Now measure the distance of the focus of the reflected pencil from the mirror, and calculate f from the formula.

The position of the image formed in any of these methods may be found in either of the following ways:—

(1) Let the image be formed as distinctly as possible on a screen, moving the screen about till the best position is found. The screen is then in the position of the true image, and its distance from the mirror must be measured.

(2) Place a pointer or a stretched wire at about the place where the image is formed. Look at the pointer and the image together, and move the head from side to side; then if they appear to move with reference to each other, they are not in the same plane, and the pointer must be moved till no such motion is seen. It is then in coincidence with the image, and its distance from the mirror must be measured.

The Optical Bench or Bank.—This is an apparatus for making optical measurements such as those we have just described. It consists of a long base fitted with a graduated scale, and along which may be slid, in a groove or otherwise, stands carrying various supports adapted for carrying lights, screens, mirrors, and other optical apparatus. Each of these stands has a mark, which, by moving over the divided scale, serves to indicate the distance through which the stand may be moved. The supports should all be capable of some lateral and vertical adjustment on their stands, so that such points as may be desired of the objects that they carry may be brought into the same straight line, parallel to the bench.

A convenient method of measuring the distance between two objects, such as screen and mirror, on the bench, is to have a straight pointer of known length carried on one of the stands, and set parallel to the bench, and so that its points are in the line with the objects whose distance is required. Move the stand carrying the pointer till its points in turn touch the objects, taking the readings of the stand. The distance required is the difference of these readings plus the length of the pointer.

EXAMPLES.

1. Show geometrically that with the help of two plane mirrors an image of a given straight line can in general be formed to coincide with any other straight line of the same length in a plane with the given one;

and that there is an infinite number of pairs of positions of the two mirrors for producing the required result.

2. Two plane mirrors make an angle, β, with each other. A ray of light travelling towards their intersection, and in a plane at right angles to it, makes an angle, α, with the mirror on which it is first incident. Show that the condition that the direction of the ray may be completely reversed by n reflexions at the mirrors—that is, that it may travel parallel to its old direction, but in the opposite sense—is—

$$2\alpha + (n - 1)\beta = \pi$$

or—

$$n\beta = \pi$$

according as n is odd or even.

3. Light coming from a point 16 cms. in front of a concave mirror falls normally on its surface, and is then brought to a focus at 5·2 cms. from the mirror: find the focal length of the mirror.

4. A concave mirror of 5 feet focal length is used to form an image on a screen which is 20 feet from it: find how far from the mirror the object must be placed.

5. A concave mirror is used to bring a pencil of light to a focus at a point, O. A convex mirror is then interposed to intercept the light normally, so that O is 4 ins. behind its surface; and the reflected pencil then comes to a focus at a point which is $6\frac{3}{4}$ ins. in front of this mirror: what is the focal length of the convex mirror?

6. The letter L is drawn on a sheet of paper and held upright before a concave mirror: describe, with drawings, the appearance that will be presented for all positions of the object.

7. If in the last question a convex mirror is used, describe the appearance that will be presented.

CHAPTER III.

REFRACTION. LENSES.

When light passes out of one transparent medium into another it does not, as a rule, continue its path in straight lines in passing across the boundary surface of the media. If the light strikes the surface normally, it will continue to travel normally through it; that is, in this case it will suffer no deviation. If the light strikes the surface obliquely, a part of it will, for almost any two media, be deviated at the surface, so as to travel in the second medium, if it is homogeneous, in straight lines, but in straight lines which are not the productions of those which formed the paths of the rays in the first medium. This action on light is called **refraction**. Another part of the light will be reflected at the surface, in accordance with the ordinary laws of reflexion.

This can be illustrated by many simple experiments. Observe

a small object under the surface of water, looking obliquely to the surface. Let a straight stick be held with one end near the eye, and so as to touch the object with the other end. It is found that one must look, not along the stick, but above it; for the light from the object, which reaches the eye, does not travel straight to the eye in the direction of the stick, but pencils which leave the object in a direction nearer to the vertical than the stick is, on reaching the surface are bent away further from the vertical, and so reach the eye. The figure shows the path of the rays which leave the visible point P, and are thus bent at the surface to reach the eye at E.

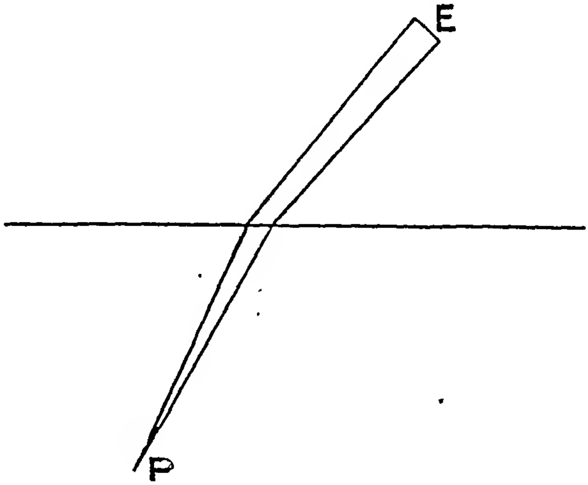


FIG. 32.

If a coin be placed in the bottom of a basin, and the eye held so that the coin is just hidden by the rim, then, on pouring water into the basin, the coin will come into view again. For the pencils by which the coin would have been seen at first, before pouring in the water, must have gone straight to the eye; but after the water is poured in light reaches the eye from the coin which first goes more upwards, and at the surface is bent away from the vertical; thus getting round the rim of the basin. The accompanying figure shows the passage of the rays in this case.

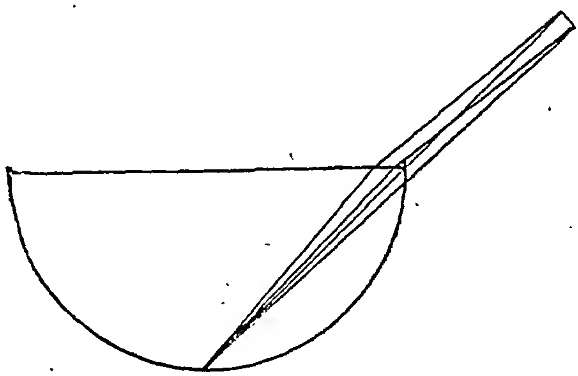


FIG. 33.

The appearance of a stick held obliquely, and partly in the water, is another example. The stick appears to be bent upwards at the surface, and to be shorter than it really is. For the pencils of light which reach the eye from it seem to

have come from points higher up than those from which they really come.

We shall now consider the precise manner in which light is refracted at the boundary surface of two media. We shall see that refraction depends on the nature of the media, being different in this respect from reflexion.

A ray of light falling on the surface is called, as in the case of reflexion, an **incident ray**; and the plane containing this ray and the normal to the surface at the point of incidence is called **the plane of incidence**, and the angle between the incident ray and the normal **the angle of incidence.** A ray going on into the second medium from the point of incidence is called a **refracted ray**; and the plane containing this ray and the normal is called **the plane of refraction,** and the angle between the refracted ray and the normal **the angle of refraction.**

The **Laws of Refraction** are these—

I. *The incident ray, the normal, and the refracted ray are in the same plane.*

II. *For the same two media, the sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction.*

The second law is generally known as **the law of sines;** it was first given by Snell, a Dutchman, but is frequently attributed to Descartes.

We shall, in general, have to consider the case of light refracted from air into a given medium, or *vice versâ.*

When light is refracted from a vacuum into a given medium, the constant ratio of the sines is called the **refractive index,** or **index of refraction,** of the medium. This quantity is generally denoted by the symbol μ. We shall see that the refractive index of a given medium differs for different colours; but, at present, and unless the contrary is specified, we shall suppose all the light considered to be of the same colour, so that μ is constant for a given medium.

When light is refracted from air into the medium, the ratio of the sines is practically the same as when it comes from a vacuum; that is, the air has practically no influence. If, then, i and r denote the angles of incidence and refraction from air or from a vacuum into a given medium, we have the relation—

$$\sin i = \mu \sin r.$$

The apparatus shown in the figure (Fig. 34) may be used to demonstrate the laws of refraction for liquids. There are two tubes, G H, L K, moving on a vertical circle, and always

directed to its centre, and provided with apertures as for the laws of reflexion. (Or the tubes may be provided with lenses, as the figure shows.) Instead of using the vertical circle the tubes may be prolonged with arms, in which are small holes at P, P′, these being equidistant from the centre of rotation. C O is a horizontal scale, which may be moved vertically, and the zero of which is vertically below the axis through the centre about which the tubes turn. C D E F is a vessel to contain the liquid, turning with the tube G H, and having a bottom, D E,

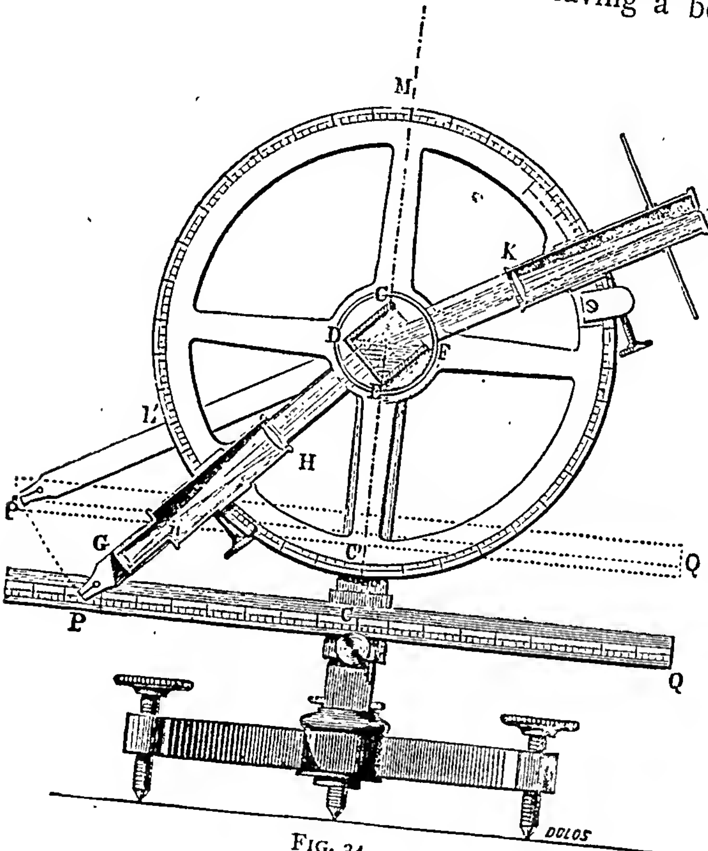


FIG. 34.

of plate glass set at right angles to G H, so that light crossing D E normally is undeviated. The surface of the liquid is adjusted to be level with the centre of the circle. K L is set in any position, and the reading through P′ taken. G H can be set to receive the light that passes through K L, and is refracted at the surface of the liquid, and, when it is so set, the reading through P is taken after suitably adjusting the scale. The ratio of C′ P′ to C P, which is the same as the ratio of the sines of the angles of incidence and refraction at the surface of the liquid, should be found constant.

Fig. 35 shows how the apparatus may be used to verify the law for solids. A B C is a half-cylinder of glass, or other transparent substance, set with the plane face A C horizontal, and the curved sides at right angles to the vertical circle. D E F K is a block of the same substance, with the face E F

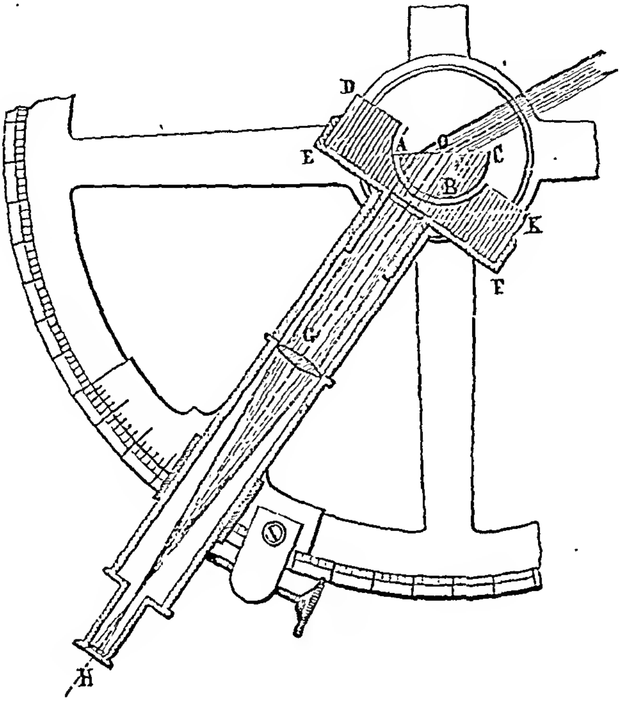


FIG. 35.

set at right angles to the tube G H, and having a cylindrical hollow into which A B C just fits. With these arrangements it is clear that the apparatus may be used in just the same way as before.

The laws of refraction, as those of reflexion, do not depend for their demonstration on such direct proofs as these, but on the consistency of the results obtained by assuming their truth as the basis of measurements that can be made with great accuracy.

When light is refracted from a medium *a* into a medium *b*, the constant ratio of the sines of the angles of incidence and refraction is called the **relative index of refraction from** *a* to *b*. This we may denote by ${}_a\mu_b$. There are some important relations existing between relative refractive indices, which we shall now consider.

It is known by experiment that the path of a ray of light is reversible. That is, if a ray along a given line in one

medium is refracted along a second line in another medium, then, when a ray passes along this latter line in the second medium, but in the opposite sense, it will emerge along the first line in the first medium. Thus if i and r are corresponding angles of incidence and refraction from medium a to medium b, r and i will be corresponding angles of incidence and refraction from b to a.

Since ${}_a\mu_b = \frac{\sin i}{\sin r}$, and ${}_b\mu_a = \frac{\sin r}{\sin i}$, it follows that ${}_b\mu_a = \frac{1}{{}_a\mu_b}$.

The following is an experiment in support of the above statement. If a plate of glass, that is, a portion with plane parallel faces, be held between the eye and a distant object, the direction of the rays falling on the glass and of those leaving it will be the same, whatever the obliquity to those rays at which the glass is held. The distant object will appear to be slightly displaced by an amount depending on the thickness of the glass and its obliquity. But the thinner the glass taken, the less will this displacement be, and the glass may be taken so thin as to make the displacement imperceptible to the eye. The figure shows the passage of the ray for this case. i and r are the angles of incidence and refraction at the first surface, at the point A. r is thus also the value of the angle of incidence, internally, at B, since the normals at A and B are parallel; and since the rays P A and B Q are parallel, the angle of refraction at B must be equal to i.

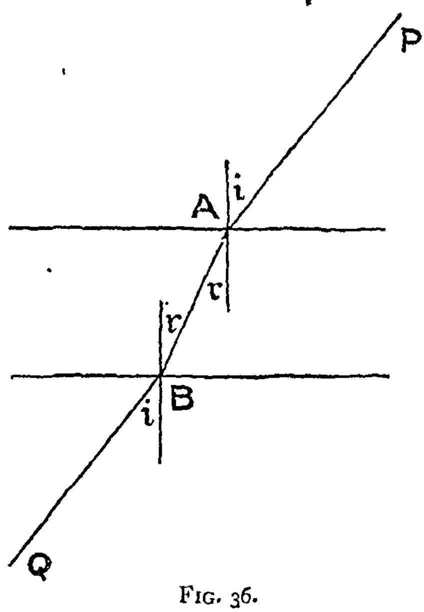


FIG. 36.

Suppose that a ray of light passes from a medium, a, through two plates of b and c, having a common face, and then emerges into a. It is found that the rays P A and C Q in the two portions of a are parallel. Now, let the angles of incidence and refraction be as shown in the figure. Then we have—

$${}_a\mu_b = \frac{\sin i}{\sin r}; \quad {}_b\mu_c = \frac{\sin r}{\sin r'}; \quad {}_c\mu_a = \frac{\sin r'}{\sin i}.$$

Thus—

$${}_a\mu_b \; {}_b\mu_c \; {}_c\mu_a = 1.$$

Or—

$$_{a}\mu_{c} = {_{a}\mu_{b}}\,{_{b}\mu_{c}}.$$

In the same way, experiment would show that any number of plates placed as indicated would not deviate a ray of light; so that we should have general relations such as the above existing among any number of relative refractive indices.

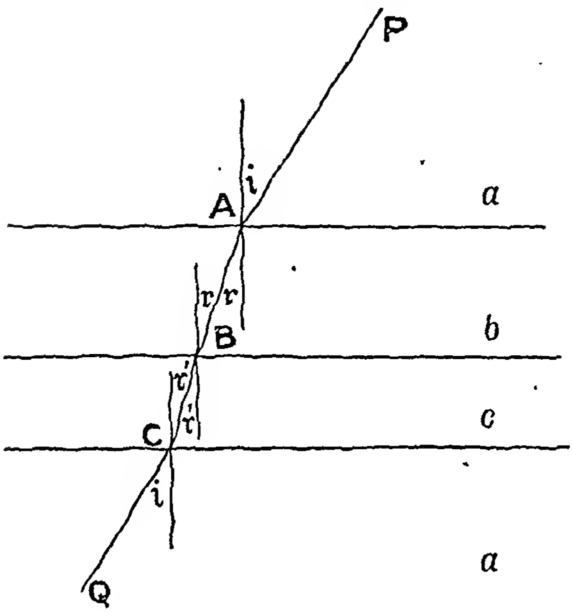


FIG. 37.

We may write the general relation for a number of media, $a, b, c, \ldots k, l$—

$$_{a}\mu_{b}\,{_{b}\mu_{c}} \ldots {_{k}\mu_{l}}\,{_{l}\mu_{a}} = 1.$$

As a particular application, suppose we have determined the refractive indices *from air into glass* and into water; call these μ, μ'. Then the refractive index from glass into water is $\frac{\mu'}{\mu}$.

Let one medium be a vacuum; denote it by v. Then we have—

$$_{v}\mu_{a}\,{_{a}\mu_{b}}\,{_{b}\mu_{v}} = 1.$$

This we may write—

$$_{v}\mu_{b} = {_{v}\mu_{a}}\,{_{a}\mu_{b}}.$$

Thus *the absolute index of refraction of b is found by multiplying the relative index of refraction from a to b by the absolute index of refraction of a.*

In this case let a be air, and b some transparent medium, such as water or glass. We have said that we may consider $_{a}\mu_{b}$ and $_{v}\mu_{b}$ to be practically the same. This is because the correcting factor $_{v}\mu_{a}$, the index of refraction from vacuum to air, or *absolute index of refraction of air*, is found by experiment to be nearly equal to unity—about 1·0003.

In general, when light passes from a rarer to a denser medium, as from air to water, the rays are bent towards the normal—the refractive index is greater than unity. When the light passes from the denser to the rarer medium, the

light is bent away from the normal towards the surface of separation.

Suppose light to pass from a denser to a rarer medium (as when it comes up to the surface of still water and emerges into the air). Let μ be the index of refraction, which in this case is less than unity. For any angle of incidence, i (measured inside the denser medium), we calculcate the angle of refraction, r, by the relation $\sin r = \frac{\sin i}{\mu}$. Thus r is always greater than i in this case.

Fig. 38 shows how a series of rays, O A, O B, O C, O D, coming from a point O, are refracted at the surface. Any value of i will give a possible value of r as long as it gives $\sin r$ less than 1; that is, as long as $\sin i$ is less than μ. Thus for all angles of incidence, from zero up to a certain value, for

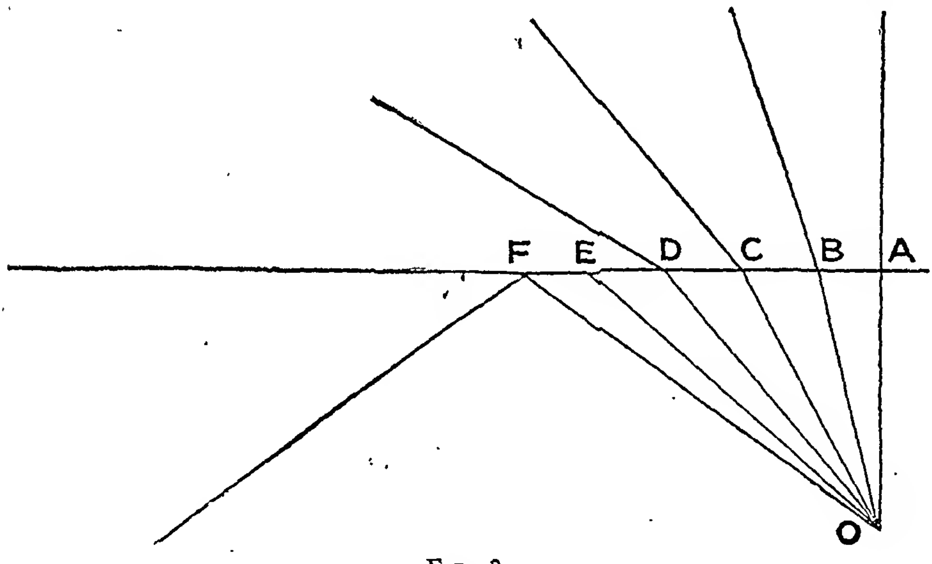


FIG. 38.

which $\sin i = \mu$, some of the light which reaches the surface is refracted into the second medium. Some will also be reflected back into the first medium. As i increases, the refracted rays get closer and closer to the surface. For the ray O E making an angle with the normal whose sine is μ, light emerges glancing along the surface. The question arises—What happens when the incident rays are beyond O E, as O F, making greater angles with the normal? In this case none of the light is refracted, but all which falls on the surface is reflected back into the first medium. This phenomenon is called **total internal reflexion.**

Suppose the second medium to be air. Then the limiting

ray O E makes with the normal an angle whose sine is equal to the index of refraction of the first medium. This quantity is denoted by $\frac{1}{\mu}$ above, since μ was taken to be the index of refraction from the first medium to the second. This angle is called the **critical angle** for the medium. If θ is the critical angle and μ the refractive index of a medium, we have—

$$\sin\theta = \frac{1}{\mu}.$$

The direction of a ray refracted from one medium to another may be found in practice by the following graphic method, when the index of refraction is known: Let X X denote the surface of separation of the two media. And let P O be a ray in the first medium incident on the second at the point O. With O as centre describe two circles with radii O A, O B in the ratio $1 : \mu$. Let OP cut the first of these in S; and draw T S M perpendicular to X X, meeting the second circle in T. Then T O produced is the path of the ray in the second medium. For—

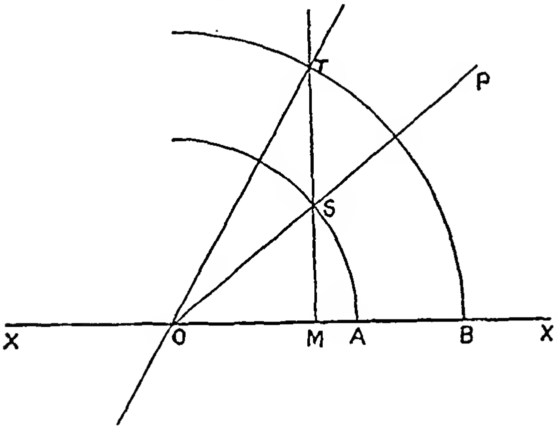


FIG. 39.

$$\frac{\sin OSM}{\sin OTM} = \frac{OT}{OS} = \mu.$$

And O S M is equal to the angle of incidence at O. Therefore O T M is equal to the angle of refraction.

If in this construction μ is < 1, O B is $<$ O A. In this case S M may or may not meet the second circle. If it does not, there is no refraction, and we have total internal reflexion. The limiting case is where S M touches the second circle at M. Then we have for the sine of the angle of incidence, which is now the critical angle, $\frac{OB}{OS}$; *i.e.* μ.

To illustrate total internal reflexion: if the eye is held under the level of the surface of water in a glass, much more light is seen reflected from the under surface than if it be looked

at from above. If a rod is placed in the glass so as to be partly immersed, the image of it formed by reflexion in the under surface is as clear as that seen directly through the water.

In looking from above at an object under the surface of water, the object may be made to appear as close to the surface of the water as we please by holding the eye near enough to the surface. As the eye approaches indefinitely near to the surface, the pencils by which the object is seen meet the under surface at angles which approach to the critical angle, and emerge more and more nearly along the surface.

The rays which come from the outside to an eye under the surface of the water, as at O, must all make in the water angles with the vertical less than the critical angle. Thus this eye will see all objects above the surface as if they were contained in a cone with the eye at the vertex, and of semi-vertical angle equal to the critical angle. All these objects will appear to be raised up, and those nearer to the surface the more so.

As the first example of the formation of an image by refraction, we shall consider the following:—

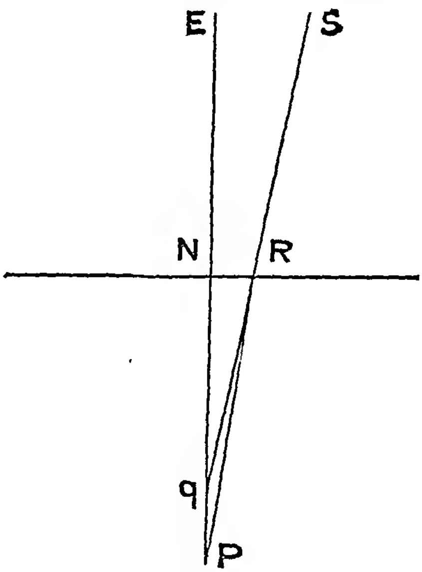


Fig. 40.

Image of Visible Point formed by Refraction at a Plane Surface.—Suppose the point P, inside a transparent medium of index μ, to be seen by an eye at E, so that E N P is normal to the surface of the medium. Draw the ray P R close to the normal; and let it be refracted along R S. Produce S R to meet P N in q. Then the image of P, since it is seen by a very narrow pencil along P N E, will be at the ultimate position of q when P R is indefinitely close to P N. Now R P N, R q N are equal to the angles of incidence (from inside) and refraction (outwards) at R. Thus we have—

$$\sin \text{RPN} = \frac{1}{\mu} \sin \text{R}q\text{N};$$

$$\frac{\text{RN}}{\text{RP}} = \frac{1}{\mu} \cdot \frac{\text{RN}}{\text{R}q};$$

$$\text{RP} = \mu \text{R}q.$$

In the limit this gives for the position of the image Q—

$$NQ = \frac{NP}{\mu}.$$

We have supposed the eye to look normally to the surface; but we shall see that, if this is not the case, no true image of P will be seen.

The result here found gives us a method of finding the refractive index of a thick plate of glass. With a reading microscope travelling vertically—that is, a microscope with a scale by which its vertical displacement can be measured—a reading is taken on a mark made on a horizontal piece of glass set so as to receive the plate. This mark corresponds to P. The plate is now put on, and a reading is taken on the image of this mark, which corresponds to Q; and another on the top of the plate, which corresponds to N. From these we can find the thickness, and the apparent thickness of the plate; and the ratio of the former to the latter is the refractive index.

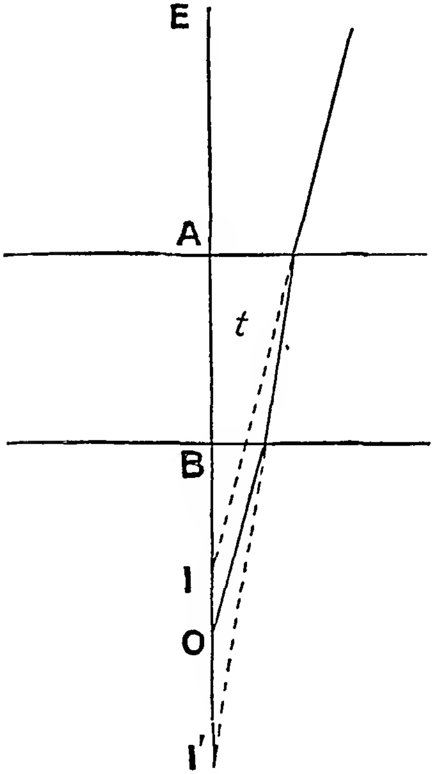


FIG. 41.

A similar method may be used to find the refractive index of a liquid.

Displacement produced by Plate looked through normally.—Let the object O (Fig. 41) be looked at through the plate of thickness t, normally to the plate. Let O be at distance d from the near face of the plate. Let μ be the refractive index of the plate.

O forms, by the first refraction at B, an image at I′, so that $BI' = \mu d$. I′ forms, by the second refraction at A, an image, I, so that—

$$AI = \frac{AI'}{\mu} = \frac{\mu d + t}{\mu} = d + \frac{t}{\mu}.$$

Thus $BI = d + \frac{t}{\mu} - t$; and $OI = t\left(1 - \frac{1}{\mu}\right)$.

Thus at whatever distance the object is from the plate, it appears to be displaced towards it by a distance $t\left(1 - \frac{1}{\mu}\right)$.

Shifting of Ray produced by Plate.—Let A B C D (Fig. 42) be a ray passing through a plate of thickness t. Draw C E perpendicular to A B produced. C E is the amount of shifting.

$$CE = CB \sin (i - r)$$
$$= \frac{t \sin (i - r)}{\cos r}.$$

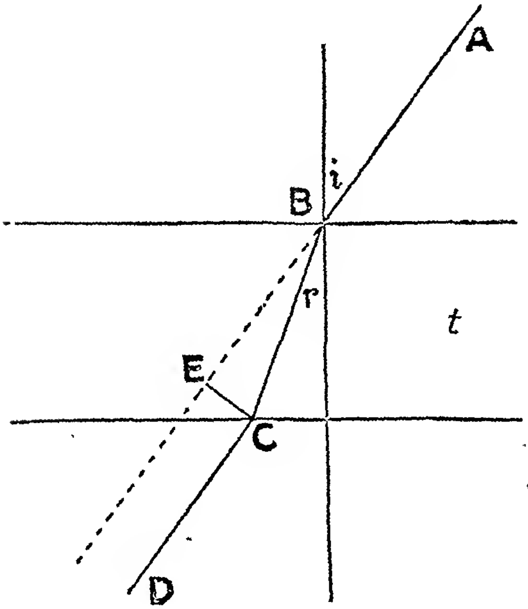


FIG. 42.

Refraction through a Prism.—In geometry a prism is a solid figure contained by planes (forming its faces) which are all parallel to one straight line, or the intersections of which (forming the edges of the prism) are all parallel. The number of the faces may be any whatever, and the ends may be any whatever; or, indeed, the prism is not usually regarded as having ends at all. For optical

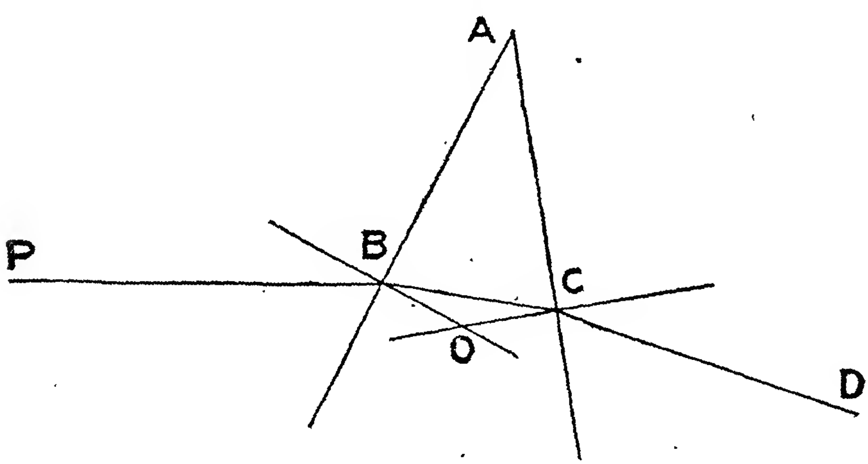


FIG. 43.

purposes the prism will be taken of triangular section. The ends are generally perpendicular to all the faces; and, in general, only two of the faces are employed, so that only

these two need be polished surfaces. The edge where these two faces meet is called the **refracting edge** of the prism. We may call these two faces the refracting faces. The dihedral angle contained by these faces is called the **refracting angle** of the prism. Any section of the prism made by a plane perpendicular to its faces is called a **principal section.** So that all principal sections are equal in all respects and parallel. And the angle made by the traces of the refracting faces in a principal section is equal to the refracting angle. Consider a ray of light in the plane of a principal section striking one face of a prism so that it emerges from the other face. It will continue in the same plane on entering the prism and on emerging. The figure represents the path of the ray. If ϕ_1, ϕ_2 are the deviations at the first and second faces, the entire deviation δ is $\phi_1 + \phi_2$.

Let μ be the refractive index of the prism. Let i_1, i_2 be the angles which the ray makes with the normals at B and C outside the prism; and r_1, r_2 the angles it makes with these normals inside the prism. We have—

$$\sin i_1 = \mu \sin r_1,$$
$$\sin i_2 = \mu \sin r_2,$$
$$\phi_1 = i_1 - r_1;\ \phi_2 = i_2 - r_2;\ \delta = i_1 - r_1 + i_2 - r_2.$$

The angles i_1, r_1, i_2, r_2 may be called the angles of external incidence, internal refraction, internal incidence, and external refraction. Now suppose the prism set so that the angle of external incidence is i_2. Then the angle of internal refraction is r_2; and since the sum of the internal angles is constant, as may easily be seen by the figure, the angle of internal incidence is r_1, and the angle of external refraction is i_1. Thus the deviation is the same as before, and the prism now occupies just the same position with regard to the incident and emergent light as it occupied before with regard to the emergent and incident light respectively. These two positions may be called conjugate. Notice that the deviations for two conjugate positions are equal. The deviation for any third position whatever would be different from that produced in these, as experiment would show, or as the above-written equations would show.

Now, there is one position of the prism which is its own conjugate—the position in which the prism is symmetrical with regard to the incident and emergent light, or in which $i_1 = i_2$; $r_1 = r_2$; and the triangle A B C is isosceles. For a given direction of the incident ray, P B, it is clear that a pair of

conjugate positions would be found by turning the prism to opposite sides of this position, one being formed by increasing i_1 and the other by diminishing i_1. Thus, as the prism is turned from this symmetrical position, the deviation either increases, whichever be the sense of turning, or else it decreases, whichever be the sense of turning. This symmetrical position, then, is the position giving either the minimum deviation of the ray or the maximum deviation. Experiment shows that *in the symmetrical position the deviation of the ray is a minimum.*

This is a very important result, and may also be proved by means of the equations—

$$\sin i_1 = \mu \sin r_1,$$
$$\sin i_2 = \mu \sin r_2,$$
$$\delta = i_1 + i_2 - r_1 - r_2.$$

From the figure $r_1 + r_2$ is supplementary to the angle at O, and therefore equal to the refracting angle, A, of the prism, Thus to see whether δ becomes a maximum or a minimum, we must see whether $i_1 + i_2$ becomes a maximum or a minimum.

Now consider the equation—

$$\sin i = \mu \sin r \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let di and dr be corresponding small increments in the angles i and r, so that—

$$\sin (i + di) = \mu \sin (r + dr) \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2), we get—

$$2 \cos \left(i + \frac{di}{2} \right) \sin di = 2\mu \cos \left(r + \frac{dr}{2} \right) \sin dr,$$

or, making di and dr indefinitely small—

$$\cos i \, di = \mu \cos r \, dr$$

—a result which comes from (1) by the differential calculus, immediately.

$$di = \frac{\mu \cos r}{\cos i} dr$$
$$= \frac{\mu \cos r}{\sqrt{1 - \mu^2 \sin^2 r}} dr.$$

The square of the coefficient of dr is—

$$\frac{\mu^2 \cos^2 r}{1 - \mu^2 + \mu^2 \cos^2 r} = \frac{1}{1 - \dfrac{\mu^2 - 1}{\mu^2 \cos^2 r}}.$$

Now, the greater r is, the less is $\mu^2 \cos^2 r$, the greater is $\frac{\mu^2 - 1}{\mu^2 \cos^2 r}$; the less is the denominator of the above fraction, and the greater is the fraction.

Thus a given small variation in the direction of the ray in the denser medium causes a greater variation in that of the ray in the other medium, the greater the angle which the first ray makes with the normal. Or, again, the variation of r from r to $r + x$, where x is a small positive angle, causes a greater increase in i than the variation of r from $r - x$ to r, or than the decrease in i caused by the variation of r to $r - x$. This result is interesting in itself.

To apply it to the case before us. Suppose r_1 and r_2 each to have the value $\frac{A}{2}$, as is the case for the symmetrical position. Then a small increment in r_1 must be accompanied by an equal decrement in r_2; and the resulting increment in i_1 is greater than the resulting decrement in i_2. So that a variation from the symmetrical position such as to increase i_1 causes $i_1 + i_2$, and therefore also the deviation, to increase; and the same would follow from a variation in the opposite sense. Thus the deviation in this position is a minimum. This position is generally called **the position of minimum deviation.**

Expression for Refractive Index of Prism.—Suppose the refracting angle A of the prism, and the angle of minimum

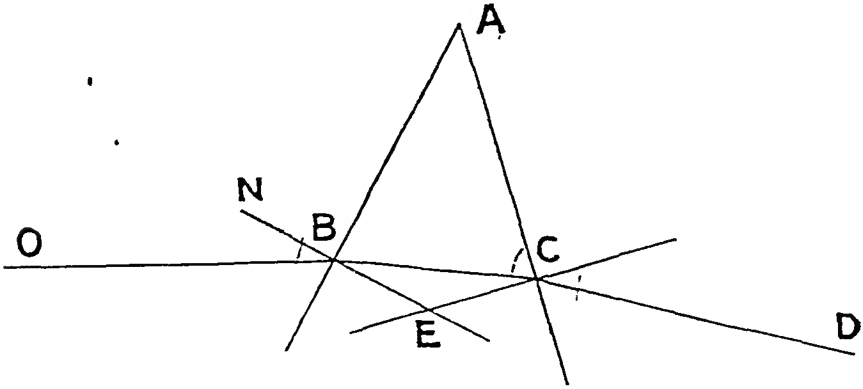


FIG. 44.

deviation, D, are known. Let the figure represent the position of minimum deviation, O B C D the ray. Then the angles i_1, i_2 are equal, and so are r_1, r_2. The triangles ABC, EBC are both isosceles. And $\mu = \frac{\sin \text{OBN}}{\sin \text{EBC}}$.

Now, the deviation at B is half the entire deviation.

$$\therefore \frac{D}{2} = OBN - EBC;$$

$$OBN = \frac{D}{2} + EBC.$$

And EBC + ECB = supplement of E = A.

$$\therefore EBC = \frac{A}{2};$$

$$\mu = \frac{\sin \frac{D+A}{2}}{\sin \frac{A}{2}}.$$

This is a very important expression for the refractive index, and is employed in determining it in practice, by the most accurate method.

When the angle of the prism is very small, we can obtain a useful expression for D. For we then have—

$$\mu = \text{limiting value of } \frac{\sin \frac{D+A}{2}}{\sin \frac{A}{2}}$$

$$= \frac{D+A}{A}.$$

Thus $D = A(\mu - 1)$.

Consider a small divergent pencil refracted through a prism, very near to its edge, in the plane of a principal section, and at minimum deviation.

The divergence of the pencil at right angles to the plane of principal section is unaltered on passing out of the prism. For consider a ray in the prism at right angles to the edge, but not in the principal section. The position of this ray determines, and determines in precisely the same manner, the positions of the entering and emerging rays. Thus, since the ray we have considered inside the prism is situated in just the same way with regard to the two faces, the entering and emerging rays are situated in just the same way on the two sides of the prism, and the ray on emerging makes the same angle with the plane of principal section as on entering. Thus the divergence at right angles to this plane is unaltered.

The divergence in the plane of principal section is

unaltered. For since the direction of the pencil is such as to give minimum deviation, a small alteration in its direction will produce no alteration, practically, in the deviation. Thus two rays in the plane of principal section will have the same divergence on emergence as on entry, the limiting ratio of the angle between the rays on emergence to the angle between them

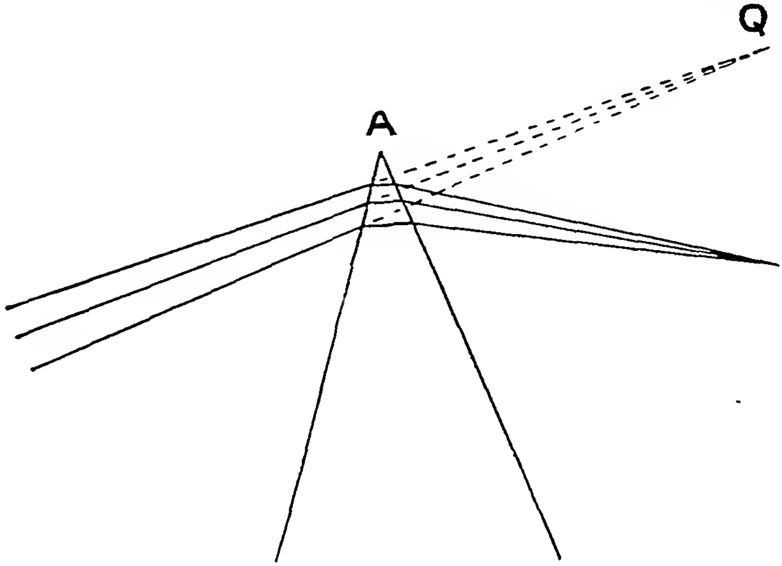


FIG. 45.

on entry, when the latter becomes indefinitely small, being unity. Thus the divergence in the plane of principal section is unaltered.

It follows that the pencil emerging from the prism will be a pencil diverging apparently from a single point in the plane of principal section, and this point, Q, is at the same distance from the prism where the rays enter it, or from A, as is the point P, from which they originally come.

Thus a visible point, or a small object, at P, gives, by means of a pencil or pencils in the position of minimum deviation, a true virtual image at Q. It should be noticed, as in the cases of the mirror, that all the rays of a large pencil from P would not pass through Q, this point being only the limiting position of the intersection of these rays when they are taken indefinitely near to each other and to the ray in the position of minimum deviation.

It could be shown in just the same manner that a pencil converging to Q, that is, so as to give a virtual object at Q, would form, after refraction, a real image at P.

The points P and Q are **conjugate foci** with respect to the prism.

LENSES.

A **lens** is a portion of a refracting medium whose boundary surfaces are, in general, portions of surfaces of revolution having a common axis: this axis is called the axis of the lens.

If the surfaces do not meet, we may suppose the boundary of the medium to be completed by a portion of a cylinder having the same axis as the lens, but these portions will have nothing to do with the ordinary action of the lens.

We shall consider lenses having for boundary surfaces portions of spheres. The axis of such a lens is the straight line joining the centres of the spheres of which the boundary surfaces are parts.

One of the bounding surfaces may be a plane, as a particular case of a sphere having an infinitely large radius. Then the axis of the lens is the straight line drawn from the centre of the spherical surface perpendicular to the plane.

The six general forms of lenses are shown in the accom-

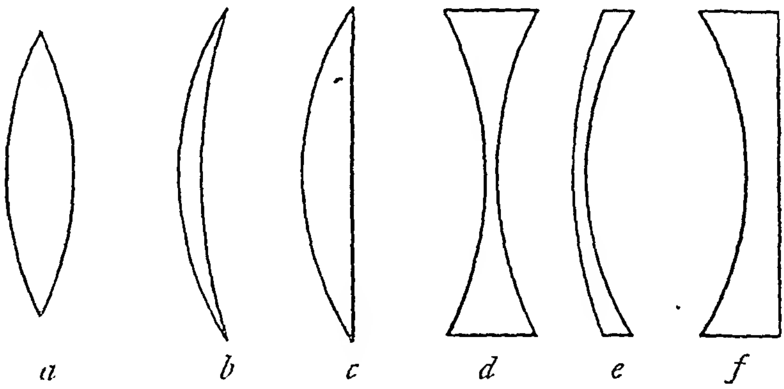


FIG 46.

panying figure. They are sometimes called by the following names :—

a, double convex ;
b, convexo-concave, concavo-convex, or meniscus ;
c, convexo-plane, or plano-convex ;
d, double concave ;
e, convexo-concave, or concavo-convex ;
f, concavo-plane, or plano-concave.

The order of the words combined to name the lens refers to the order in which the light falls on the two surfaces. The surface on which the light first falls is called the *front*, and the other the *back* surface.

A more important classification is as follows :—

Lenses **thickest** in the middle are called **convex**, or **convergent**.

Lenses **thinnest** in the middle are called **concave**, or **divergent**.

To lead up to the refraction of light through lenses, let us first consider the following case : Suppose a small pencil of light to fall on the spherical boundary of a refracting medium, along a radius of the sphere : to find the focus of the pencil after refraction.

We shall make the same convention with regard to signs as was made in the case of reflexion. Distances will be measured along a radius from the surface of the medium, and those measured in the direction opposite to the incident light will be considered positive; those measured in the other direction, negative.

Let the boundary surface be represented by the circular arc A R in the figure, its radius A O being r, so that r is, in this

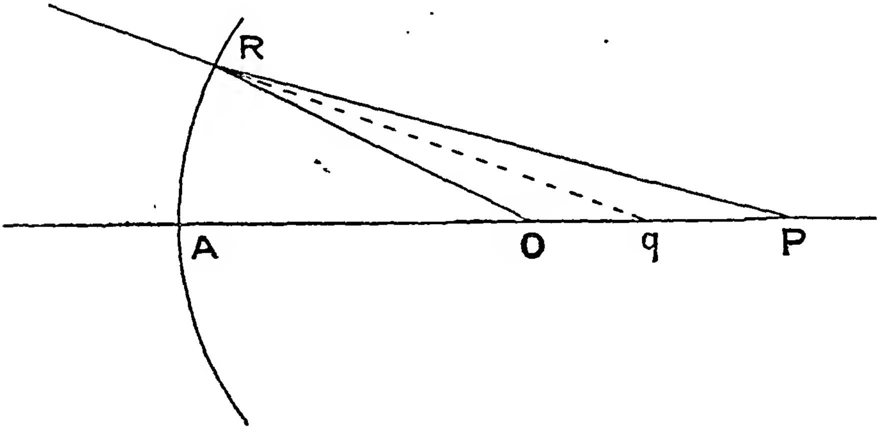


FIG. 47.

figure, positive. The refracting medium, of index μ, is to the left of the surface.

P is a visible point on the radius through A. Let a ray from P meet the surface in R.

O R is normal to the surface at R. Thus the refracted ray at R lies in the plane O R P. And, supposing μ greater than 1, its production backward lies between R O and R P. Let this ray meet A O P in q.

As the ray P R approaches to the line P A, and ultimately coincides with it, q moves up to a definite limiting position: call this Q. Let $AP = u$, $AQ = v$. These distances are both positive in the figure, so that u and v are both positive quantities.

We have from the figure the relation—

$$\sin ORP = \mu \sin ORq.$$

To get a relation among the lengths of lines, we write this—

$$\sin ROP \cdot \frac{OP}{RP} = \mu \sin ROq \cdot \frac{Oq}{Rq}.$$

$$\text{Thus } \frac{OP}{RP} = \mu \frac{Oq}{Rq}.$$

In the limit this becomes—

$$\frac{OP}{AP} = \mu \frac{OQ}{AQ}.$$

Or—

$$\frac{u-r}{u} = \mu \frac{v-r}{v};$$

$$v(u-r) = \mu u(v-r).$$

Divide throughout by uvr, and we get—

$$\frac{1}{r} - \frac{1}{u} = \mu\left(\frac{1}{r} - \frac{1}{v}\right).$$

Or, we may write—

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}.$$

In the case we have taken u, v, r are all positive; but the formula would be found to be universally true, whatever be the signs of these quantities. Let us, for example, suppose

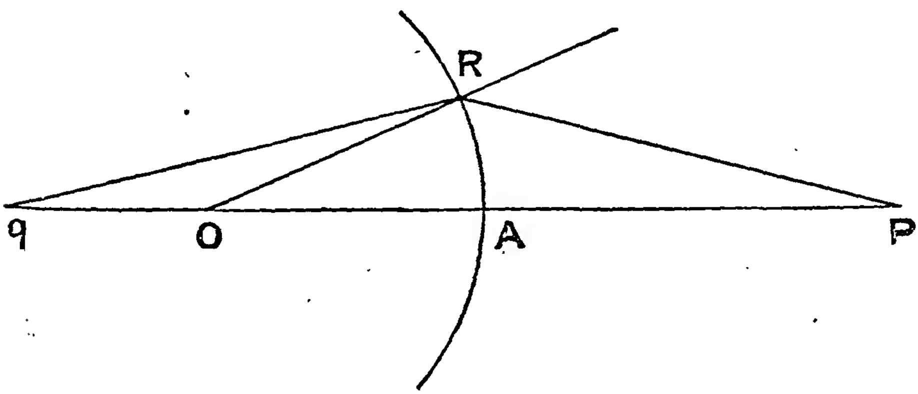


FIG. 48.

the bounding surface of the medium to be convex. Then it may be seen that the refracted rays may meet the radius through P on either side of A. Suppose they meet it on the left of A, which is the case for positions of P far from A.

Since PRO is supplementary to the angle of incidence, and qRO is the angle of refraction, the figure gives—

$$\sin \text{PRO} = \mu \sin q\text{RO}.$$

$$\text{Thus } \sin \text{ROP} \cdot \frac{\text{OP}}{\text{RP}} = \mu \sin \text{RO}q \cdot \frac{\text{O}q}{q\text{R}};$$

$$\frac{\text{OP}}{\text{RP}} = \mu \frac{q\text{O}}{q\text{R}}.$$

In the limit this becomes—

$$\frac{\text{OP}}{\text{AP}} = \mu \frac{\text{QO}}{\text{QA}}.$$

In this, of course, the mere absolute magnitudes of the lengths involved are meant.

Now—

$$\text{OP} = \text{OA} + \text{AP} = -r + u,$$

since $-r$ is a positive quantity, and r is in numerical magnitude equal to OA.

$$\text{AP} = u,$$
$$\text{QO} = \text{QA} - \text{OA} = -v + r;$$
$$\text{QA} = -v.$$

Therefore the above relation becomes—

$$\frac{u - r}{u} = \mu \frac{-v + r}{-v} = \mu \frac{v - r}{v}.$$

This is the same relation as before, and gives, as before—

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}.$$

If a small pencil of light along A O in the medium has Q for its focus, it will, on emergence, have P for its focus; since the paths of the rays are reversible. P and Q may be called **conjugate foci.**

The above formula is useful in itself; and would be used to find, for instance, the apparent position of a point, when we know the real position, or *vice versâ*, the point being embedded in a refracting medium with a spherical boundary, and looked at normally to the surface. Thus we can optically determine the thickness of a lens if we know its refractive index, and the curvature of one surface, and if we can find the position of the image of a mark on the other surface where the axis meets it when looked at through the first surface along the axis.

The formula has, however, a more important application, as we now proceed to show.

Suppose we have a lens whose thickness may be neglected. Let the radii of its front and back surfaces be r and s, regard being had to the sign. In the figure drawn both r and s are

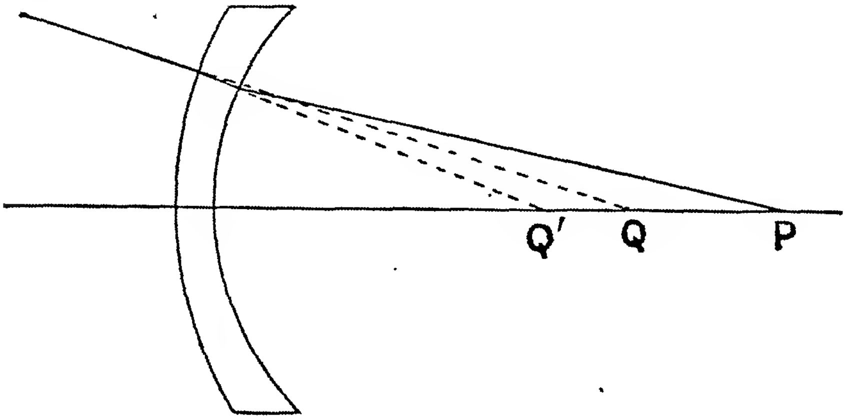


FIG. 49.

positive. Suppose there is a small axial pencil with focus at P, a point at a distance, u, from where the axis meets the lens. Since the thickness of the lens is neglected, we may suppose the axis to meet it in a point; and distances will be measured along the axis from this point. Let this pencil give rise, on entering the lens, to a pencil with focus Q′, at distance v' from the lens; and let this, on emerging, give rise to a pencil with focus at Q, at distance v from the lens.

Then from refraction of the pencil into the lens we have—

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And for the refraction out of the lens, since the corresponding refracting index is $\frac{1}{\mu}$, we have—

$$\frac{\frac{1}{\mu}}{v} - \frac{1}{v'} = \frac{\frac{1}{\mu} - 1}{s}.$$

Or—

$$\frac{1}{v} - \frac{\mu}{v'} = \frac{1 - \mu}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Adding (1) and (2), we get—

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right).$$

This very important formula gives the relation between the positions of a point on the axis of a lens and of its image as formed by the lens.

From the reversibility of the paths of the rays, it follows that a pencil on the left-hand side of the lens, having Q for focus, would give a pencil on the other side, with P for focus. The same thing would follow from the formula. The points P and Q are conjugate foci with respect to the lens.

We have met with four cases of conjugate foci. These were formed with respect to (1) spherical mirrors, (2) prisms, (3) refracting spherical surfaces, (4) lenses. In any case the existence of the conjugate foci is inferred at once from the reversibility of the rays. In any case a pair of conjugate foci has the following property: Suppose a pencil with P as focus, real or virtual; let this give a pencil with Q as focus, real or virtual; then a pencil with Q as focus of the same sort as that we have just supposed to be at Q will give a pencil with P as focus of the same sort as we supposed at P. Thus, for example, take the case of the prism, Fig. 45. A pencil with real focus at P gives a pencil with virtual focus at Q. A pencil coinciding with this latter, but convergent, whereas the latter was divergent, having a virtual focus at Q, will give rise to a pencil with real focus at P. A pencil with real focus at Q would give, by refraction through the prism, something quite different; in fact, a pencil with no point-focus at all.

Suppose that P (Fig. 49) goes to infinity, that is, that the pencil incident on the lens is a parallel one. The corresponding value of v is called the focal length of the lens. It is denoted by f, and is given by—

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right).$$

The corresponding position of Q is called the principal focus of the lens, and we shall, in general, denote it by F.

The formula for the lens may now be written—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

For a *convex* lens, thickest in the middle, by examining all the cases that may occur, we see that $\frac{1}{r}$ is always algebraically less than $\frac{1}{s}$. For example, in the accompanying figure of a meniscus, with the light incident on the convex side, r and s

are both negative, and r is numerically smaller than s. Therefore $\frac{1}{r}$ is numerically greater and algebraically less than $\frac{1}{s}$. And

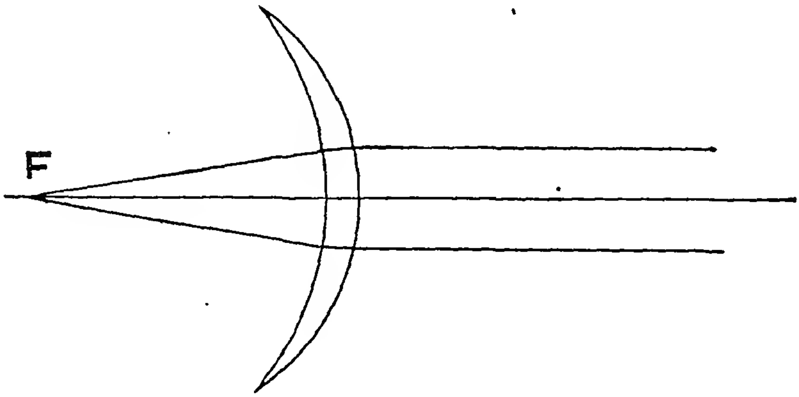


FIG. 50.

we should find $\frac{1}{r}$ algebraically less than $\frac{1}{s}$ in any case of a convex lens.

Thus $\frac{1}{r} - \frac{1}{s}$ is a negative quantity;

$$\therefore f \text{ is } \textit{negative};$$

and F is on the *negative side* of the lens.

The rays which the parallel pencil gives on passing through the lens *converge* to the point F, and really pass through it.

Thus F is a *real principal focus.*

From the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

since f is negative, it follows that $\frac{1}{v}$ is always algebraically less than $\frac{1}{u}$. Let us consider the cases that can occur with this condition. The figures show them.

(1) u positive; v may be positive; v must then be greater than u (Fig. 51).

(2) u positive; v may be negative (Fig. 52).

(3) u negative; v must be negative, and numerically less than u (Fig. 53).

In any case it is seen that the pencil is more convergent (or less divergent) after passing through the lens. Thus the lens is called a *converging lens.*

For a *concave* lens, thinnest in the middle, by examining

all the cases that may occur, we see that $\frac{1}{r}$ is always algebraically greater than $\frac{1}{s}$.

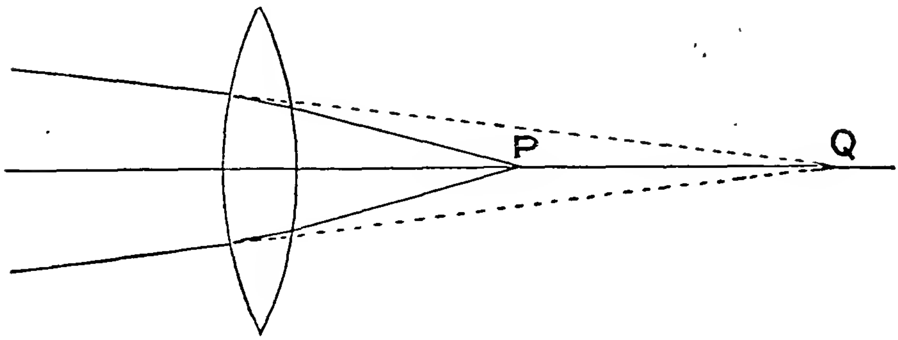


FIG. 51.

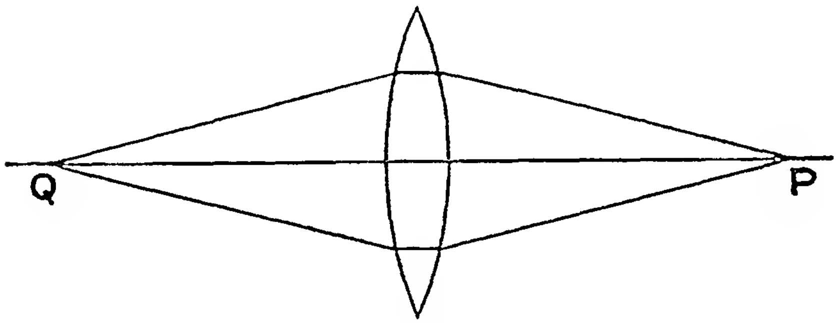


FIG. 52.

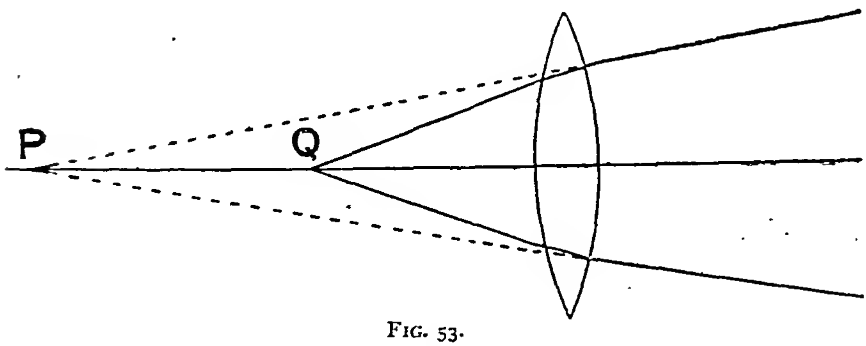


FIG. 53.

Thus $\frac{1}{r} - \frac{1}{s}$ is a positive quantity;

$\therefore f$ is *positive;*

and F. is on the *positive side* of the lens.

The rays which the parallel pencil gives on passing through the lens appear to diverge from the point F, but do not really pass through it (Fig. 54).

Thus F is a *virtual principal focus.*

From the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

since f is positive, it follows that $\frac{1}{v}$ is always algebraically

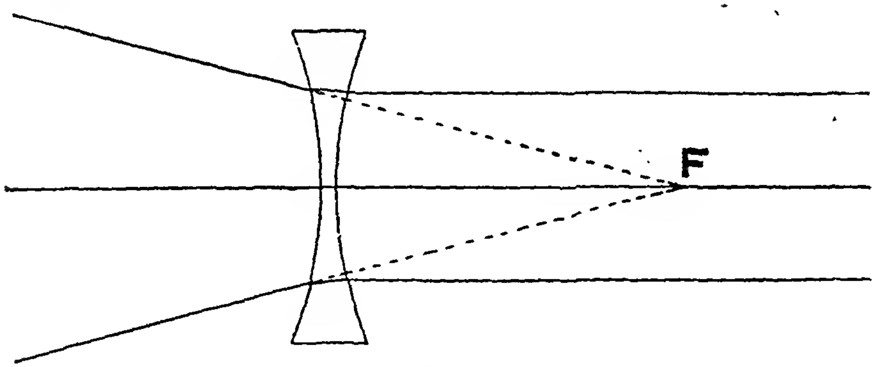


FIG. 54.

greater than $\frac{1}{u}$. Let us consider the cases that can occur with this condition. The figures show them.

(1) u positive; v must be positive, and numerically less than u (Fig. 55).

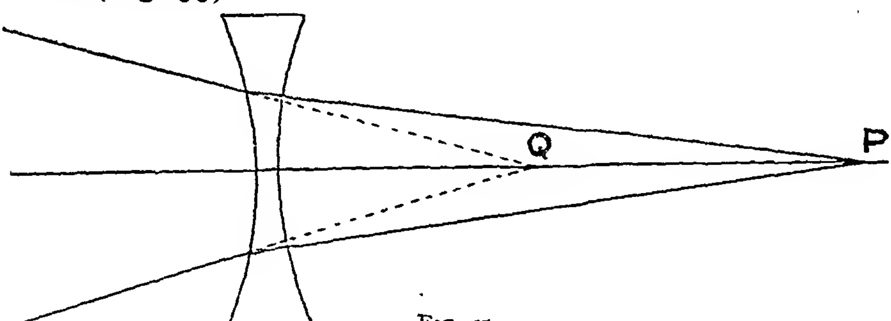


FIG. 55.

(2) u negative; v may be positive (Fig. 56).

(3) u negative; v may be negative; v must then be numerically greater than u (Fig. 57).

In any case it is seen that the pencil is more divergent (or less convergent) after passing through the lens. Thus the lens is called a *diverging lens.*

It should be noticed that a convex lens behaves like a concave mirror in having a real principal focus, and in converging axial pencils; a concave lens behaves like a convex mirror in having a virtual principal focus and in diverging axial pencils.

Notice again, however, that both concave mirror and concave lens have positive focal lengths; and convex mirror and convex lens have negative focal lengths, according to our convention with regard to sign.

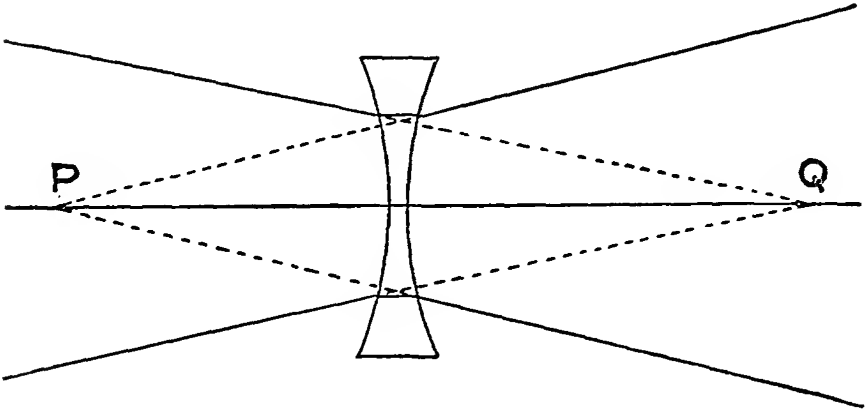


FIG. 56.

Any lens will have two principal foci, one for light falling on it from each side. They will be at equal distances from the lens. For if the radii of the surfaces for light falling on the lens from one side—right, say—are r and s, the radii for

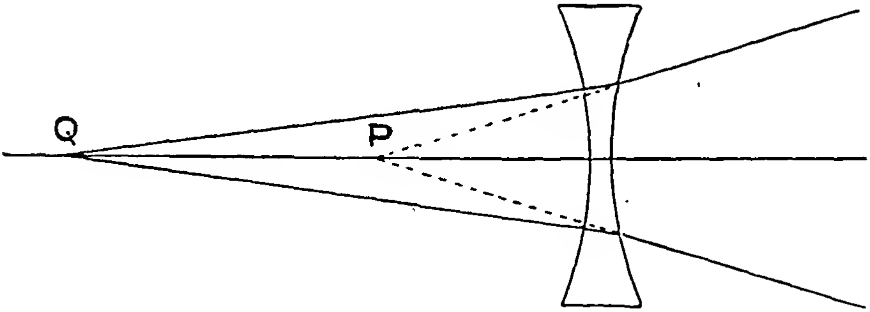


FIG. 57.

light falling on the lens from the left will be $-s$ and $-r$; so that the focal length, f', for light from the left is given by—

$$\frac{1}{f'} = (\mu - 1)\left(\frac{1}{-s} - \frac{1}{-r}\right)$$
$$= (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{1}{f}.$$

We shall denote the principal focus for light coming from the negative side by F′.

It should be noticed that a principal focus, on account of

the reversibility of rays, has also this property. A pencil of light having a principal focus of a lens for its focus will, after passing through the lens, become a parallel pencil, along the axis. Notice that the pencil must have the principal focus for a real or a virtual focus according as it is a real or a virtual focus of the lens.

We shall now consider the variations in position of the image as the position of the object is changed. We shall make use of the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

or—

$$\frac{1}{v} = \frac{1}{f} + \frac{1}{u}.$$

1. *Convex lens*—

When $u = \infty$, $v = f$; *i.e.* object at infinity gives image at F.

As u decreases in value from ∞ to the positive quantity $-f$, v decreases algebraically from f to $-\infty$. Thus, as object moves up from infinity to F′, image moves off from F to infinity on the left.

Object at F′ gives image at infinity.

As u decreases from $-f$ to 0, v decreases from ∞ to 0. Thus, as object moves from F′ up to the lens, the image, which is virtual, moves from infinity up to the lens.

It should be noticed that the object gives a real or a virtual image according as it is beyond or within the principal focus F′.

2. *Concave lens*—

When $u = \infty$, $v = f$; *i.e.* object at infinity gives virtual image at F.

As u decreases from ∞ to 0, v decreases from f to 0. Thus, as the object moves up from infinity to the lens, the image moves up from F to the lens.

It should be noticed that the image is always virtual.

We have only considered the cases in which the object is real; and the general conclusions drawn refer only to such cases. The cases in which the object is virtual may be considered in just the same way as the others by the help of the formula.

Centre of Lens.—Consider a lens of thickness t, so that AB $= t$ in the figure. Suppose a ray to pass through the lens so as to be undeviated. Then it must undergo equal and opposite deviations at entry and emergence, at the points

R and S. The radii to the surfaces at R and S must thus be parallel. Let these be O_1R, O_2S. SR lies in the plane with

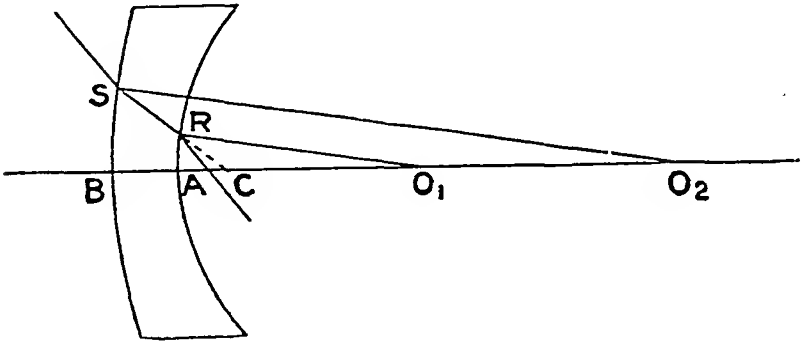


FIG. 58.

them, and so will meet the axis, say in the point C. C is a fixed point, as we shall show.

Let $AO_1 = r$; $BO_2 = s$. The two figures $RACO_1$, $SBCO_2$, are similar. Thus—

$$\frac{AC}{AO_1} = \frac{BC}{BO_2};$$

$$i.e. \ \frac{AC}{r} = \frac{AC + t}{s};$$

$$\therefore AC(s - r) = rt;$$

$$AC = \frac{rt}{s - r};$$

$$BC = \frac{st}{s - r}.$$

These values will always be found for the distances of C from A and from B measured to the right, whatever be the signs of r and s.

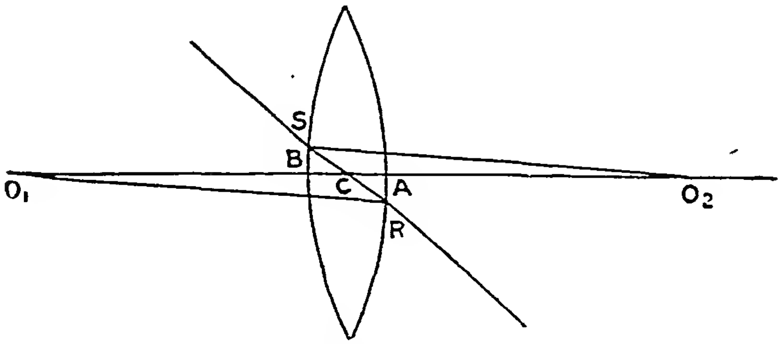


FIG. 59.

Take, for example, the double convex lens drawn in the above figure.

As before, we have—

$$\frac{AC}{AO_1} = \frac{BC}{BO_2};$$

$BC = t - AC$; $AO_1 = -r$; $BO_2 = s$.

$$\therefore \frac{AC}{-r} = \frac{t - AC}{s};$$

$$AC \cdot s = -rt + AC \cdot r;$$

$$AC = -\frac{rt}{s - r};$$

Thus the distance of C to the right of A is $\frac{rt}{s-r}$ (a negative quantity, since r is negative, s and t positive), and $BC = \frac{st}{s-r}$.

The point C is called *the centre of the lens.*

If we neglect the thickness, t, of the lens, $AC = BC = 0$. Thus, for a thin lens the centre is *the point* at which the axis meets the lens; and a ray through this point will be undeviated.

An object situated on the axis of a lens, and all in a plane

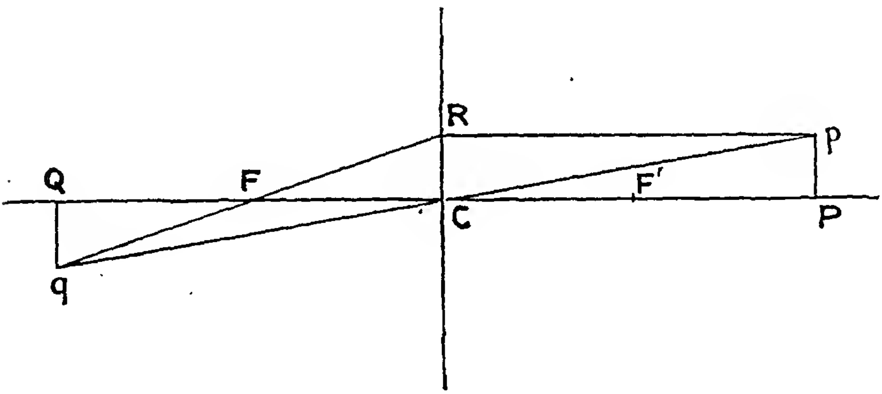


Fig. 60.

at right angles to the axis, and of very small dimensions, will have an image of itself formed on the axis, and similar to itself.

Consider a convex lens, and such an object Pp, P being on the axis, and farther off than F′. The point p sends out a ray, pR, parallel to the axis, which passes, after leaving the lens, through the focus F, very nearly, because p is very near the axis. The ray pC, through the centre C, goes on undeviated. The point of intersection, q, of these two rays gives the focus of all the rays, which come from P, after passing

through the lens. Draw qQ perpendicular to the axis. Then—

$$\frac{QF}{FC} = \frac{Qq}{CR} = \frac{Qq}{Pp} = \frac{QC}{CP}.$$

Thus Q is a fixed point, and the images of all points of the object are on the same plane through Q. And since—

$$\frac{Qq}{Pp} = \frac{QC}{CP},$$

it follows that the image is similar to the object.

It must be noticed that, for these results to hold, any point, as p, of the object must be so near the axis that the ray from it parallel to the axis may practically pass through F after deviation by the lens.

Graphic Construction of Images.—We shall now consider a method, similar to that used in the case of mirrors, of constructing the image, formed by a thin lens, of a small object on the axis. As in that case, we shall find the focus of a small pencil near the axis, after it passes through the lens by drawing two of its rays. And we can draw the three following rays :—

(1) The ray which passes through the centre of the lens : this continues in the same straight line.

(2) The ray which comes in parallel to the axis : this is continued through the focus F.

(3) The ray which comes in through the focus F′ : this is continued parallel to the axis.

We shall now consider the three different general cases

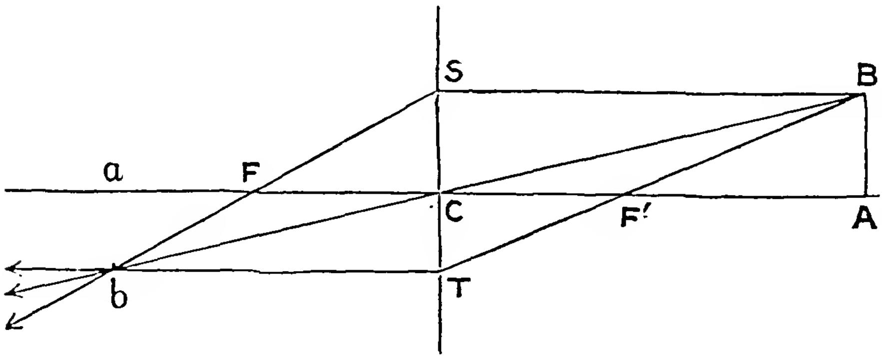


FIG. 61

that may occur, giving the graphic construction for each case, the object being real.

I. *Convex Lens: Object beyond focus* F'.

Take A B, a small object in a plane perpendicular to the axis, with the point A on the axis (Fig. 61).

The rays B C, B S, B T, from the point B of the object, are respectively through the centre, parallel to the axis, and through the focus F'. They continue through the centre, through the focus F, and parallel to the axis. They give by their intersection the point *b*, the image of B. *b* is seen by means of a pencil of rays of which these are three. The image *a b* of the whole object is shown in magnitude and position by drawing *b a* perpendicular to the axis.

The image is *real* and *inverted.*

II. *Convex Lens: Object between lens and focus* F'.

The same rays as before are here drawn. The pencil they give, after passing through the lens, has now a virtual focus, *b.*

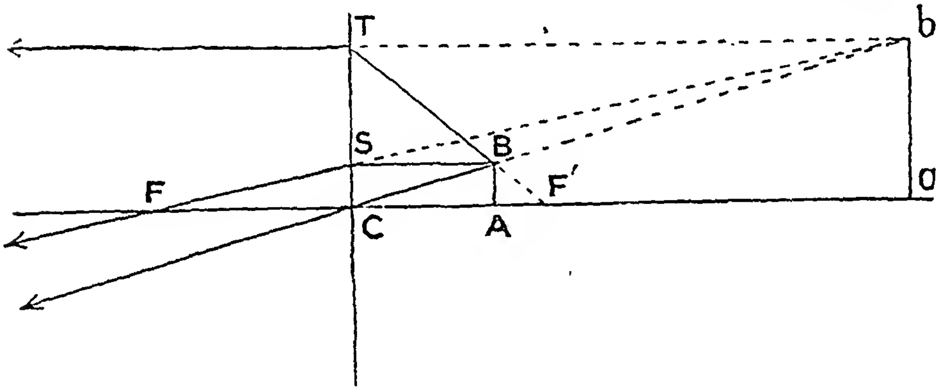


FIG. 62.

The rays *b* B, *b* S, *b* T have now no physical existence; but the action on the left-hand side of the lens, and the effect on the eye, are the same as if they had. The whole image *b a* is found as before.

The image is *virtual* and *erect.*

III. *Concave Lens.*

The construction follows the same lines as before. Notice, now, the interchange of positions of F and F'. Another figure should be drawn in which A is between C and F. It will be seen that there is no radical difference (Fig. 63).

The image is *virtual* and *erect.*

Remarks may be made on these cases similar to those made for mirrors.

The image is now real or virtual according as it is on the opposite side or on the same side of the lens as the object. In this respect the images formed by a mirror and lens differ.

For with a *lens* the rays from a given point pass really through the image of the point when, and only when, the image is on the *other* side of the lens; with a *mirror* they pass really

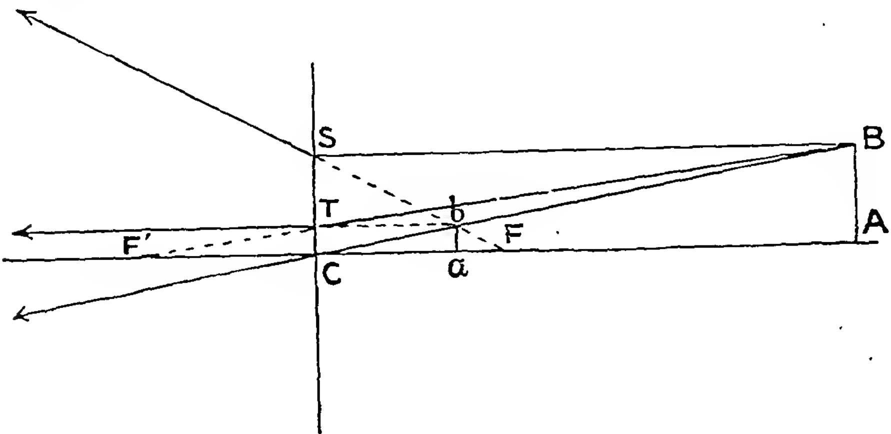


FIG. 63.

through the image when, and only when, the image is on the *same* side.

The following statements have been made for mirrors, and apply to mirrors and lenses :—

The image is erect or inverted according as it is on the same side or on the opposite side of C from the object.

The image is real when inverted, virtual when erect.

Magnification.—This term has just the same signification here as for a mirror. Denote the magnification by m, as before. Then from any of the figures just drawn—

$$m = \frac{ab}{AB}.$$

We shall now deduce some important expressions for m. What follows is applicable to any one of the three figures. All three should be referred to.

From similar triangles *abc*, ABC, we get—

$$m = \frac{ab}{AB} = \frac{Ca}{CA};$$

$$i.e.\ m = \frac{\text{distance of image from lens}}{\text{distance of object from lens}} \quad . \quad . \quad (1)$$

Taking the trace of the lens on the paper to be a straight line at right angles to the axis, as is done in the case of the mirrors, we get the following expressions :—

From similar triangles abF, CSF, we get—

$$m = \frac{ab}{AB} = \frac{ab}{CS} = \frac{aF}{CF}$$

$$i.e.\ m = \frac{\text{distance of image from focus F}}{\text{focal length}} \quad . \quad . \quad (2)$$

From similar triangles CTF′, ABF′, we get—

$$m = \frac{ab}{AB} = \frac{CT}{AB} = \frac{CF'}{AF'}$$

$$i.e.\ m = \frac{\text{focal length}}{\text{distance of object from focus F}'} \quad . \quad (3)$$

The numerical values merely of the quantities in these fractions are to be understood.

We can express m in symbols in any of the following ways, from the above expressions, the numerical values of the fractions given being understood :—

$$m = \frac{u}{v} = \frac{v-f}{f} = \frac{f}{u+f}.$$

The expression number 3, involving the position of the object, and not of the image, is about the most important.

We shall draw some general inferences with regard to the magnification produced.

Convex Lens.—When the object is at a distance from the lens greater than $2f$, it is at a greater distance than f from F′. Thus by (3) the image is *diminished.* When the object is at a distance from the lens less than $2f$, it is at a less distance than f from F′. Thus the image is *magnified.*

Concave Lens.—The object is always at a greater distance than f from F′; so that the image is always *diminished.*

We shall now enumerate, for reference, the characteristics of the image formed by either lens, and for all positions of the object, with regard to the following four particulars :—

(1) Position ;
(2) Whether real or virtual ;
(3) Whether erect or inverted ;
(4) Size.

Convex Lens—

Object at greater distance from lens than $2f$; image to left of F, real, inverted, diminished.

Object at distance $2f$; image at distance $2f$ on left, real, inverted, equal to object.

Object at distance between f and $2f$; image to left of F, real, inverted, magnified.

Object between lens and F′; image to right of lens, virtual, erect, magnified.

Concave Lens—

Object in front of lens; image in front of lens, between lens and F, virtual, erect, diminished.

Throughout, the close resemblance of behaviour between a concave mirror and a convex lens, and between a convex mirror and a concave lens, should be noticed.

Combination of Thin Lenses in Contact.—Suppose any number of lenses of unappreciable thickness be placed so as to have the same axis and to be all in contact. Let the lenses be n in number, and have focal lengths $f_1, f_2, \ldots f_n$. Suppose a small object, at distance u from the combination, forms an image in the first at distance v_1; this forms an image in the second at distance v_2; and so on; so that an image is formed in the last at distance v_n.

Then—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1};$$

$$\frac{1}{v_2} - \frac{1}{v_1} = \frac{1}{f_2};$$

$$\frac{1}{v} - \frac{1}{v_{n-1}} = \frac{1}{f_n}.$$

Therefore, by addition—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2} + \ldots + \frac{1}{f_n} = \Sigma\left(\frac{1}{f}\right).$$

Now, a single lens, if placed in the same position as the combination, would form an image of the object in the same position as this image, if its focal length, F, is given by—

$$\frac{1}{F} = \Sigma\left(\frac{1}{f}\right)$$

This lens is said to be **equivalent** to the given system of thin lenses.

If the lenses are separated by finite distances, a_1, a_2, etc., no single thin lens can be found so as in all cases to produce an image of a given small object in the same position as the lenses would. In this case the final position of the image is to be calculated by such equations as—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1},$$

$$\frac{1}{v_2} - \frac{1}{v_1 + a_1} = \frac{1}{f_2},$$

and so on.

If we have two lenses, of focal lengths f_1, f_2, separated by an interval, a; and if v is the distance from the second at which is formed a focus for *a parallel pencil* striking the first, then—

$$\frac{1}{v_1} = \frac{1}{f_1}$$

$$\text{and } \frac{1}{v} - \frac{1}{v_1 + a} = \frac{1}{f_2}$$

$$\therefore \frac{1}{v} = \frac{1}{f_1 + a} + \frac{1}{f_2}$$

$$v = \frac{f_2(f_1 + a)}{f_1 + f_2 + a}$$

This is the focal length of a lens which must be used instead of the combination, and placed in the position of the second, to bring the parallel pencil to the same focus as the combination.

Determination of Focal Length.—The focal length of a lens can be found by many methods. We shall suppose, here, that the thickness of the lens is inconsiderable.

Suppose a small pencil of light to fall on the lens axially from a focus, P, and to produce a conjugate focus, Q. Then, if the distances of P and Q from the lens can be measured, the focal length can be calculated by the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

For a convex lens this method may be practised by fixing the lens in a holder on the optical bank, with its axis parallel to the bank, and setting up a small object, such as a pair of cross-wires, on another holder, so that this object can be moved along the axis of the lens. Then set a screen on the other side of the lens and fix it in the best position for receiving an image, formed by the lens, of the object. u and v can now be measured, and in this case it must be noticed that v is negative.

For a concave lens with a real object we should always

have a virtual image. The method may be carried out by using a virtual object. Place a convex lens so as to converge a pencil of light to a point on its axis. Now place the concave lens in the way of this pencil, so that the rays do not reach this focus, but converge to one further off, being diverged by the concave lens. The distances from the lens, of these two foci, must be measured; they are respectively $-u$ and $-v$.

A concave lens may be combined with a convex of known focal length, f_1, and such that it forms with the concave a convex combination, whose focal length, F, can be found. Then the focal length, f_2, of the concave is found by—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

For a convex lens a pencil of parallel rays, such as the rays from the sun, may be allowed to fall on the lens along its axis. The point of convergence of these rays can then be found by receiving them on a screen. This point is a principal focus of the lens, and its distance from the lens is, numerically, the focal length.

In any case we may use a virtual image if we can find its position. We may, for instance, find the focal length of a concave lens by letting parallel rays fall on the lens, giving a virtual focus; and then measure the distance of this focus from the lens. The position of a virtual image may be found as follows: A convex lens and screen are arranged at a fixed distance apart, and the distance of an object from the lens, so placed as to produce an image on the screen, is found. Then, if the lens and screen are placed so as to produce an

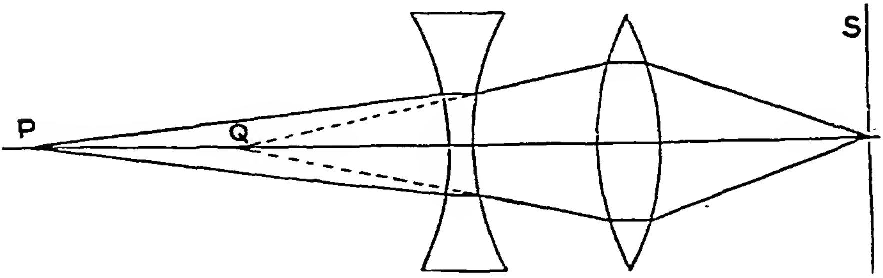


FIG. 64.

image on the screen of a given virtual image, the position of this image is known.

The focal length of the lens and its distance from the screen must be such as to allow it to get near enough to the virtual image in any case.

The accompanying diagram shows the way in which the arrangement is used to determine the position of the virtual image Q of an object, P, formed by a concave lens.

Another method, similar in principle to this, for doing the same thing, is to have, instead of convex lens and screen, a reading telescope: this consists simply of a convex lens, a pair of cross-wires, taking the place of S, at which the image is to be formed, and a lens of short focal length behind the cross-wires by which to see the image.

EXAMPLES.

1. A horizontal plate of glass 0·3 in. thick is under a depth of 2·5 ins. of water: to an eye looking vertically downwards, find how far below the surface of the water a small spot on the under surface of the glass appears to be, the refractive indices of glass and water being 1·6 and 1·3.

2. The minimum deviation of a ray of light produced by passing through a prism of angle $60^\circ\ 6'\ 20''$ is $42^\circ\ 40'\ 20''$: show how to use these results to determine the refractive index of the glass prism, and find it, having given—

$$
\begin{gathered}
\text{L} \sin 51^\circ\ 24' = 9{\cdot}89294,\ \text{L} \sin 30^\circ\ 4' = 9{\cdot}69984, \\
\text{L} \sin 51^\circ\ 23' = 9{\cdot}89284,\ \text{L} \sin 30^\circ\ 3' = 9{\cdot}69963, \\
\log 1{\cdot}5610 = 0{\cdot}19340,\quad \log 1{\cdot}5600 = 0{\cdot}19312.
\end{gathered}
$$

(Lond. Int. Sci. Hons., 1886.)

3. Taking the refractive index from air to glass as $\frac{3}{2}$, draw an accurate picture of the path of a ray of monochromatic light, which falls at an incidence of 60° on the face of a prism, whose vertical angle is 30°. (Science and Art Advanced, 1894.)

4. Find the apparent position of a small object at the centre of a globe containing water, when seen by an eye outside; the radius of the globe being 5 ins., and the refractive index of water $1\frac{1}{3}$.

5. Given a double concave lens of 5 cms. thickness, the radii of curvature of its faces being 15 and 20 cms. respectively: find the position of the image of a point on the axis 24 cms. from the nearer face. (Lond. Int. Sci. Hons., 1884.)

6. A small air-bubble in a sphere of glass 4 ins. in diameter, appears, when looked at so that the bubble and the centre of the sphere are in a line with the eye, to be 1 in. from the surface: what is its true distance? ($\mu = 1{\cdot}5$.) (Lond. Int. Sci. Pass, 1887.)

7. An object is 20 feet from a screen: given two convex lenses respectively of 9 ins. and 18 ins. focal length, explain how you will obtain (1) an erect and magnified, (2) an inverted and magnified image of the object on the screen. (Lond. Int. Sci. Pass, 1886.)

CHAPTER IV.

FOCAL LINES—ABERRATION—CAUSTICS.

We have considered, hitherto, cases of pencils reflected or refracted in such manners as to give, after reflexion or refraction, pencils with definite point-foci. We shall now see that, in general, the reflected or refracted pencil will not have a focus, that is, there is no one point through which all its rays pass, even in the limit when the pencil is taken indefinitely narrow, as there is in the special cases we have considered. These cases, however, which are the easiest to consider, are at the same time the most important in their bearing on practical applications of optics. The property of the pencil, that it has no point-focus, is called **astigmatism.** For an astigmatic pencil there are two regions, along its length, of maximum concentration of the rays, or of maximum intensity. These approximate, in the cases we shall have to consider, as the pencil is taken indefinitely narrow, to two straight lines in directions at right angles to each other and to the pencil. These lines are called **focal lines.** It will be convenient to consider one ray of the pencil as the **principal ray** round which the others are clustered. The points in which the principal ray is met by the focal lines are called the focal points. An astigmatic pencil may be illustrated by a number of stretched threads all lying close together, but not passing through a point, and made to pass through two straight slits at two different places along them, set at right angles to each other and to the general direction of the threads, care being taken that no thread is bent out of the straight line by the edge of a slit. It may happen, of course, that the focal lines of a pencil are virtual, so that they are lines through which the rays would pass if produced.

Suppose an eye so placed as to receive pencils coming from a visible point after they have been rendered astigmatic by reflexion or refraction. These pencils will give no true geometrical image of the point. And the assemblage of such pencils coming originally from the various points of an object give no true image of the object. Such a case is when an object is looked at under water obliquely to the surface, or when an object is viewed by oblique reflexion in a spherical

mirror. In such cases, as we know, a sort of image is seen; but this will be more or less blurred—the images of the various points will be situated between the focal lines of the corresponding pencils. This image will be more blurred the further apart the focal lines of each pencil are. The quantities of light coming from two neighbouring points of the object do not reach the eye as if they came from two distinct points, but will overlap and encroach upon each other, more or less, thus causing indistinctness.

Suppose a straight line, as object, giving a series of astigmatic pencils (by reflexion or refraction); and suppose the image, of the sort above mentioned, formed by them, to come out a straight line parallel to the common direction of all the focal lines, of one sort, of the pencils. Then the result is that we have a straight line (the image) from all points of which rays proceed to the eye; and there are no rays, coming originally from the object, reaching the eye that do not pass through points on this straight line. There is, therefore, formed, in this case, a distinct image of the object, with no confusion, the image coinciding with the assemblage of focal lines, of one sort, of the pencils which reach the eye. The various points of the image do not correspond, separately, with the various points of the object, so that if the various points of the object were to be distinguished from each other, since these have corresponding to them in the image short overlapping pieces, the image would be blurred; but regarding the object merely as a straight line, the image also comes out a definite straight line, and the light proceeding from it to the eye may be regarded as an assemblage of pencils proceeding from its various points as from foci, although the light in any one of these pencils did not come from a single point of the object.

The appearance presented to an eye will depend upon where the eye is placed. For instance, we have seen what the appearance is in the case of an eye looking at a small object under water normally to the surface of the water. The image thus seen is the only true geometrical image of this object formed by rays refracted through the surface, or the only true image that can be seen by an eye looking at the object through the surface. But if the object is viewed obliquely to the surface, a confused image will be seen by means of astigmatic pencils; and this image will be in a different position from the true image.

Suppose a narrow pencil, with P as focus, to fall on a plane

or spherical reflecting or refracting surface. Let P R be the principal ray of the pencil. Let P O be normal to the surface. All the rays of the reflected or refracted pencil (or deviated pencil) will meet P O. The section of this pencil by a plane

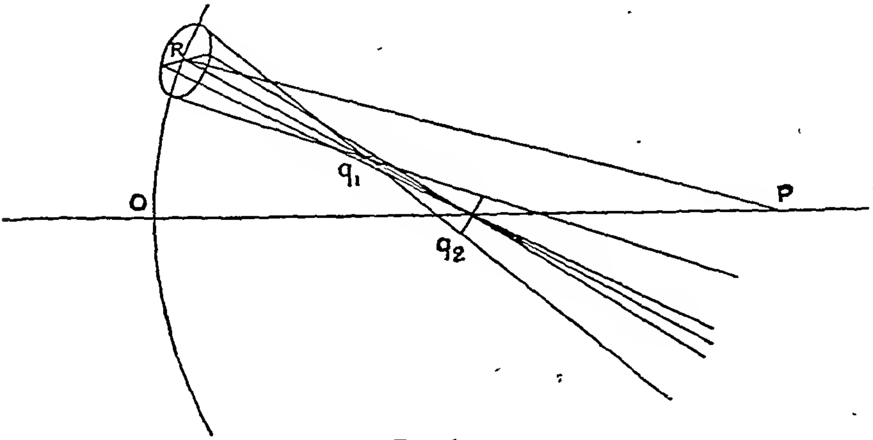


FIG. 65.

through the point where P O is met by the deviated ray from R and perpendicular to this ray is a very elongated figure of 8, approximating to a straight line as the pencil becomes very narrow. Denote it by q_2. This is called the **secondary focal line.**

All the rays of the pencil which are in any plane through P O will converge to or diverge from a point in this plane. So that all the rays of the pencil, being situated in a series of planes through P O, will converge to or diverge from a small straight line, q_1, at right angles to the plane P O R. This line is called the **primary focal line.**

The plane P O R is called the **primary plane.**

The plane through the deviated ray from R at right angles to the primary plane is called the **secondary plane.**

The primary and secondary focal lines are, then, at right angles to the primary and secondary planes.

We shall now consider, in detail, some of the pencils formed by reflexion and refraction of pencils with point-foci, and find the positions of their focal lines.

Reflexion at Plane Surface.—All the rays of a given pencil pass, after reflexion at a plane surface, accurately through the geometrical image of the focus of the given pencil. Thus no pencil will give a pencil having focal lines in this case. The reflected pencil will always have a point-focus.

Refraction at Plane Surface.—Let the pencil from P, with rays such as P R, P S, be refracted at a plane surface. The figure is drawn for refraction from a denser to a lighter medium.

Draw P O normal to the surface. The secondary focal line, q_2, is on P O.

To find the position of the primary focal line, q_1, let us consider two neighbouring rays P R, P S, in a plane with P O. Let the angles of incidence and refraction of these rays be i, r; $i + di$, $r + dr$. Let μ be the index of refraction.

Let $PR = u$; $q_1R = v_1$; $q_2R = v_2$.

Now, $\angle RPS = di$; $\angle Rq_1S = dr$.

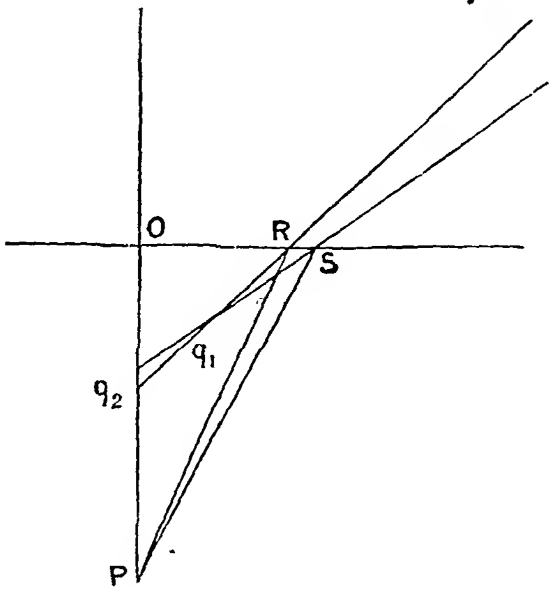


FIG. 66.

$$\text{Thus } di = \frac{RS \cos i}{u};\ dr = \frac{RS \cos r}{v_1}.$$

But since $\sin i = \mu \sin r$,

$$\therefore \cos i\, di = \mu \cos r\, dr,$$

$$\therefore \frac{\cos^2 i}{u} = \frac{\mu \cos^2 r}{v_1},$$

$$v_1 = \frac{\mu u \cos^2 r}{\cos^2 i}.$$

Again, $OR = u \sin i = v_2 \sin r$.

$$\therefore v_2 = \mu u.$$

From these results we have—

$$v_1 = v_2 \frac{\cos^2 r}{\cos^2 i}.$$

Thus q_1 is nearer to or further from the surface than q_2 according as r is greater or less than i, that is, according as the refraction is into the lighter or denser medium.

Reflexion at Spherical Surface.—Let C be the centre of the sphere.

Let CO = R; PR = u; Rq_1 = v_1; Rq_2 = v_2.
Let the angles of incidence and reflexion at R be i.

$$\angle RCS = \angle RPS + \angle PRC - \angle CSP$$
$$= \angle RPS + \tfrac{1}{2} PRq_1 - \tfrac{1}{2}\angle PSq_1.$$

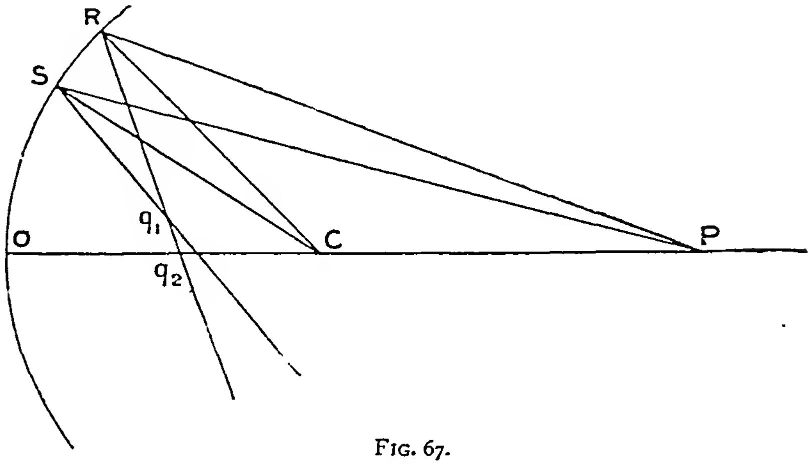


FIG. 67.

$$\text{So } \angle RCS = \angle Rq_1S + \tfrac{1}{2}\angle PSq_1 - \tfrac{1}{2}\angle PRq_1.$$
$$\therefore \angle RPS + \angle Rq_1S = 2\angle RCS;$$
$$i.e. \frac{RS\cos i}{u} + \frac{RS\cos i}{v_1} = \frac{2RS}{R};$$
$$\frac{1}{u} + \frac{1}{v_1} = \frac{2}{R\cos i}.$$

Let $\angle$ R$Cq_2 = \theta$.

$$\text{Then } \frac{R}{u} = \frac{\sin(\theta - i)}{\sin\theta};$$
$$\frac{R}{v_2} = \frac{\sin(\theta + i)}{\sin\theta}.$$
$$\therefore \frac{R}{v_2} + \frac{R}{u} = 2\cos i,$$
$$\frac{1}{v_2} + \frac{1}{u} = \frac{2\cos i}{R}.$$

Or this may be proved as follows:—

$$\triangle Rq_2P = \triangle Rq_2C + \triangle RCP;$$
$$\therefore v_2 u \sin 2i = v_2 R \sin i + Ru \sin i;$$
$$\therefore \frac{1}{v_2} + \frac{1}{u} = \frac{2\cos i}{R}.$$

Refraction at Spherical Surface.—Let the angles of incidence and refraction at R and S be i, r, $i + di$, $r + dr$.

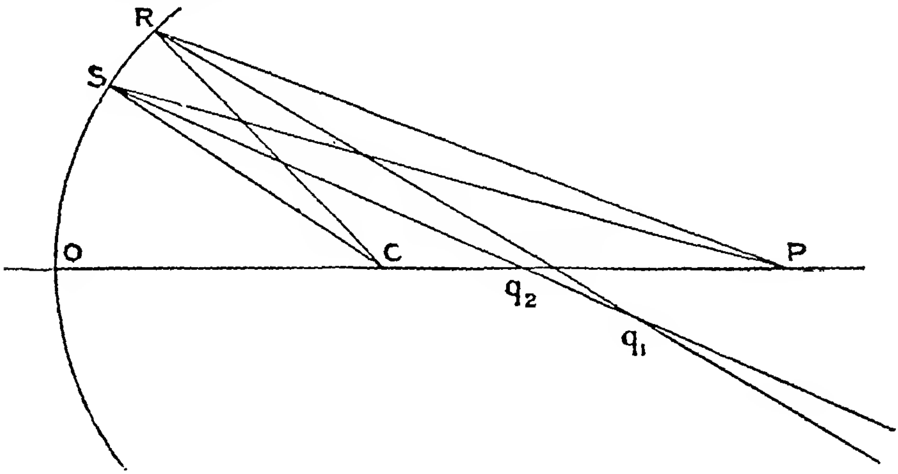


FIG. 68.

Let $CO = R$; $PR = u$; $Rq_1 = v_1$; $Rq_2 = v_2$.

$$\begin{aligned} \text{Then } di &= PSC - PRC \\ &= RPS - RCS; \\ dr &= q_1SC - q_1RC \\ &= Rq_1S - RCS. \end{aligned}$$

$$\text{Thus } di = \frac{RS \cos i}{u} - \frac{RS}{R};$$

$$dr = \frac{RS \cos r}{v_1} - \frac{RS}{R}.$$

$$\text{But } \sin i = \mu \sin r;$$
$$\therefore \cos i \, di = \mu \cos r \, dr.$$

$$\text{Thus } \frac{\cos^2 i}{u} - \frac{\cos i}{R} = \frac{\mu \cos^2 r}{v_1} - \frac{\mu \cos r}{R};$$

$$\therefore \frac{\mu \cos^2 r}{v_1} - \frac{\cos^2 i}{u} = \frac{\mu \cos r - \cos i}{R}.$$

Let $\angle RCO = \theta$.

$$\text{Then } \frac{R}{u} = \frac{\sin(\theta - i)}{\sin\theta} = \cos i - \cot\theta \sin i;$$

$$\frac{R}{v_2} = \frac{\sin(\theta - r)}{\sin\theta} = \cos r - \cot\theta \sin r.$$

Multiply the second of these equations by μ; subtract the first from it, and divide by R, and we get—

3. A box contains an equal number of crowns, shillings and pennies; the total amount in the box is £3. 13*s.*: find the number of each.

4. ₹100 is divided among an equal number of men, women and boys; each man receives ₹2. 8*a.*, each woman ₹2 and each boy ₹1. 12*a.*: find the number of men, women or boys.

5. A bag contains a certain number of rupees, twice as many half-rupees, and 4 times as many quarter-rupees: the whole sum amounts to ₹33: find the number of each.

6. Among how many children may ₹60 be divided so that each child may receive a rupee, an eight-anna piece, a four-anna piece and a two-anna piece?

87. *Example.* *A* and *B* together have ₹13. 8*a.*, *B* and *C* together have ₹8. 8*a.*, *A* and *C* together have ₹11. 8*a.*; how much has *A*?

₹13. 8*a.* + ₹11. 8*a.* = twice *A*'s money + *B*'s money + *C*'s money;

but ₹8. 8*a.* = *B*'s money + *C*'s money.

∴ (₹13. 8*a.* + ₹11. 8*a.* − ₹8. 8*a.*) or ₹16. 8*a.* = twice *A*'s money;

∴ *A*'s money = ₹16. 8*a.* ÷ 2 = ₹8. 4*a.*

Or thus:

(₹13. 8*a.* + ₹8. 8*a.* + ₹11. 8*a.*) or ₹33. 8*a.* = twice *A*'s money + twice *B*'s money + twice *C*'s money;

∴ (₹33. 8*a.* ÷ 2) or ₹16. 12*a.* = *A*'s money + *B*'s money + *C*'s money;

but ₹8. 8*a.* = *B*'s money + *C*'s money;

∴ *A*'s money = ₹16. 12*a.* − ₹8. 8*a.* = ₹8. 4*a.*

EXAMPLES. 52.

1. *A* and *B* together have ₹6. 0*a.* 3*p.*, *B* and *C* together have ₹4. 15*a.* 9*p.*, *A* and *C* together have ₹5. 15*a.*; how much has *A*?

2. *A* and *B* together have ₹24. 1*a.*, *B* and *C* together have ₹19. 15*a.*, *A* and *C* together have ₹23. 12*a.*; find how much *B* has.

3. A horse and a cow are together worth ₹101, a cow and a sheep are together worth ₹31, a horse and a sheep are together worth ₹81; find the price of a horse, of a cow and of a sheep.

4. A mark and a gulden are together worth 2*s.* $11\frac{1}{2}$*d.*, a gulden and a rouble are together worth 5*s.* $1\frac{1}{4}$*d.*, a rouble and a mark are together worth 4*s.* $1\frac{1}{4}$*d.*; find the value of a mark, of a gulden and of a rouble.

5. A man and a woman together have ₹30. 7*a.* 6*p.*, the woman and a boy together have ₹20. 8*a.*, the man and the boy together have ₹25. 9*a.* 6*p.*; find how much the man, the woman and the boy together have.

XIX. FACTORS AND PRIME NUMBERS.

88. If one number divides another *exactly*, the first is said to be a **factor** (or *sub-multiple*) of the second, and the second is said to be a **multiple** of the first. Thus 5 is a factor of 15, and 15 is a multiple of 5.

In speaking of the factors of a number we exclude the number *one* or *unity*, which may be said to be a factor of any number.

[*N. B.* In the present section the word *divisible* is used in the sense of *exactly divisible.*]

89. An **even** number is a number divisible by 2. An **odd** number is a number not divisible by 2.

90. Criteria of Divisibility:

A number is divisible

by 2 when its last figure is 0, or an *even* digit; as 310, 54:

4 when its last *two* figures represent a number divisible by 4; as 300, 320, 324:

8 when its last *three* figures represent a number divisible by 8; as 2000, 3400, 3240, 3816:

5 when its last figure is 0 or 5; as 370, 345:

10 when its last figure is 0:

3 when the sum of its digits is divisible by 3; as 126, 402:

9 when the sum of its digits is divisible by 9; as 477, 801:

11 when the difference between the sum of its digits in the *odd* places and the sum of its digits in the *even* places is either 0, or divisible by 11; as 34672, 582934.

To determine whether a number is divisible by 7, 11, or 13 we have the following rule:

Divide the figures of the number into groups containing *three* each, as far as possible, counting from right to left. Add the alternate groups, and subtract the smaller sum from the greater; then if the remainder is 0, or is divisible by 7, 11, or 13, the number itself is also divisible by 7, or by 11, or by 13.

Thus 98126 is divisible by 7, but not by 11 or by 13: for $126-98=28$ which is divisible by 7, but not by 11 or by 13.

91. If a number is divisible separately by two numbers which have no common factor, it is also divisible by their product.

If a number is divisible by 3 (or 9), any other number expressed by the *same digits* is also divisible by 3 (or 9).

focal lines undergoes refraction at the second surface, and we wish to investigate the form of the pencil so produced.

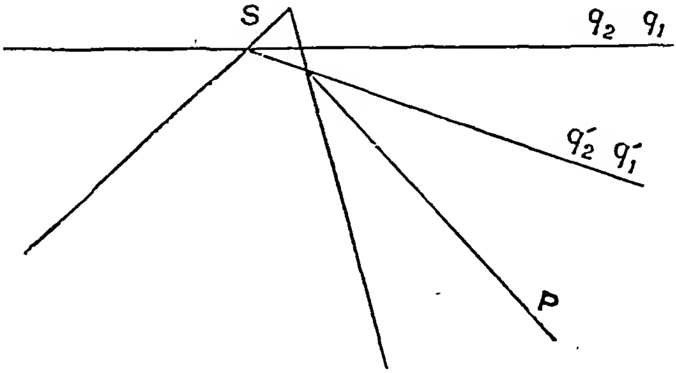


FIG. 70.

q'_2 is an assemblage of points in the plane of the paper. A pencil going from one of these points in the direction q'_2 S has, after refraction at S, a secondary focal line q_2. All the rays of the given pencil from any one point of q'_2 must, since they are in a plane at right angles to that of the paper, after refraction, meet in a point of q_2. Thus all the rays from all the points of q'_2 meet in points in the plane of the paper at q_2. Thus q_2 is the secondary focal line of our astigmatic pencil after refraction at S. Thus this focal line depends only on the position of q'_2, and not on that of q'_1; and has the same position as that of a pencil with focus at q'_2.

In the same way the rays from the various points of q'_1, which is at right angles to the plane of the paper, give, by their intersection after refraction, the various points of q_1, the primary focal line, and this is the same as that of a pencil with focus at q'_1.

We can thus find the positions of q_1 and q_2. By the formulæ at p. 79—

$$v_1 = \frac{1}{\mu} \cdot \frac{\cos^2 i'}{\cos^2 r'} \cdot v'_1$$

$$= \frac{1}{\mu} \cdot \frac{\cos^2 i'}{\cos^2 r'} \cdot \mu \frac{\cos^2 r}{\cos^2 i} \cdot u$$

$$= \frac{\cos^2 i' \cos^2 r}{\cos^2 r' \cos^2 i} \cdot u.$$

$$v_2 = \frac{1}{\mu} v_2'$$
$$= \frac{1}{\mu} \mu u$$
$$= u.$$

For the pencil at emergence to diverge from a point, we must have $v_1 = v_2$. Thus we must have $i = i'$, $r = r'$, or the pencil must be in the position of minimum deviation. In this case the distance of its focus at emergence from the edge is u. These results coincide with what we have already seen at p. 62.

Pencil passing obliquely and centrically through Thin Lens.—When a pencil passes centrically, as P R S T, that

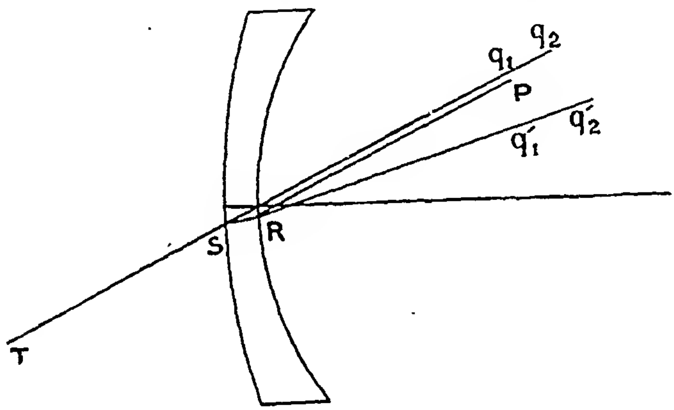


FIG. 71.

is, so that the part R S in the lens passes through the centre, it is undeviated. For a thin lens, the points R, S correspond with each other and with the centre; and we may suppose distances to be measured from any of these three points indifferently.

Let the surfaces of the lens have radii R, S; and let μ be the index of refraction. Let the pencil P R S T have P for focus, and be inclined at an angle, i, to the axis of the lens; and let r be angle of refraction into the lens. Suppose this pencil has, on entering the lens, the focal lines q'_1, q'_2; and on emergence the focal lines q_1, q_2.

Let $\mathrm{R}P = u$; $\mathrm{R}q'_1 = v_1'$; $\mathrm{R}q'_2 = v_2'$; $\mathrm{S}q_1 = v_1$; $\mathrm{S}q_2 = v_2$.

Then we have—

$$\frac{\mu \cos^2 r}{v_1'} - \frac{\cos^2 i}{u} = \frac{\mu \cos r - \cos i}{\mathrm{R}}.$$

And since the index of refraction out of the lens is $\frac{1}{\mu}$—

$$\frac{\cos^2 i}{\mu v_1} - \frac{\cos^2 r}{v_1'} = \frac{\frac{1}{\mu}\cos i - \cos r}{S}.$$

Or—

$$\frac{\cos^2 i}{v_1} - \frac{\mu \cos^2 r}{v_1'} = \frac{\cos i - \mu \cos r}{S}.$$

Adding and dividing by $\cos^2 i$, we get—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{\mu \cos r - \cos i}{\cos^2 i}\left(\frac{1}{R} - \frac{1}{S}\right).$$

Again—

$$\frac{\mu}{v_2'} - \frac{1}{u} = \frac{\mu \cos r - \cos i}{R};$$

$$\text{and } \frac{1}{v_2} - \frac{\mu}{v_2'} = \frac{\cos i - \mu \cos r}{S};$$

$$\therefore \frac{1}{v_2} - \frac{1}{u} = (\mu \cos r - \cos i)\left(\frac{1}{R} - \frac{1}{S}\right).$$

When i is small, $\cos^2 i$ is very nearly equal to unity, and the difference between v_1 and v_2 is very small. Thus the focal lines are very near together, and a good approximation to a geometrical image is obtained for points near the axis.

In this case, i and r being small quantities of the first order, $\cos i$ and $\cos r$ differ from unity by small quantities of the second order. So that we have the equations, correct to small quantities of the first order,—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{v_2} - \frac{1}{u} = (\mu - 1)\left(\frac{1}{R} - \frac{1}{S}\right).$$

Thus the distance of the image is given, to the first order of small quantities, by the ordinary formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

When a broad pencil of light, with a focus P, is reflected or refracted at a spherical surface, the rays do not, after reflexion or refraction, all pass through the same point. If the pencil is made indefinitely narrow and along the axis, that is, the radius through P of the surface, the rays do converge, after reflexion or refraction, to a definite point Q on the axis. In the same way, a broad pencil with a point P on the axis of

96. *The H. C. F. of two or more numbers is the product of all their common prime factors.*

Example 1. Find the H. C. F. of 18 and 30.

$18 = 2 \times 3 \times 3$; $\quad 30 = 2 \times 3 \times 5$.

The factors common to the two numbers are 2 and 3; hence the H. C. F. required $= 2 \times 3 = 6$.

Note. In finding the H. C. F. it is not necessary to find the prime factors of all the numbers. It is sufficient to find the prime factors of *one* of the numbers, and to form the product of those that divide each of the remaining numbers exactly.

Example 2. Find the H. C. F. of 84, 140 and 168.

Now, $84 = 2 \times 2 \times 3 \times 7$; and we find that each of the remaining numbers is divisible by $2 \times 2 \times 7$, but not by 3; therefore the H. C. F. required $= 2 \times 2 \times 7 = 28$.

EXAMPLES. 55.

Find, by the method of factors, the H. C. F. of

1. 9 and 24.
2. 20 and 48.
3. 35 and 80.
4. 126 and 144.
5. 90 and 325.
6. 252 and 348.
7. 150 and 375.
8. 256 and 788.
9. 480 and 792.
10. 15, 35, 120.
11. 16, 24, 140.
12. 90, 125, 342.
13. 224, 336, 728.
14. 625, 750, 1225.
15. 868, 3164, 4228.

97. The following rule gives the most convenient method of finding the H. C. F. of two numbers:

Divide the greater number by the less, the divisor by the remainder, then the second divisor by the second remainder, and so on, until there is no remainder; the *last divisor* is the H. C. F. required.

Example 1. Find the H. C. F. of 384 and 1296.

Process:

```
384 ) 1296 ( 3
      1152
      ----
       144 ) 384 ( 2
             288
             ---
              96 ) 144 ( 1
                    96
                    --
                    48 ) 96 ( 2
                         96
                         --
```

∴ The H. C. F. required is 48.

$$v' = \mu u + \frac{\mu^2 - 1}{\mu} \cdot \frac{y^2}{2u}.$$

This result is correct as far as the third power of y, that is, if we agree to neglect the fourth and higher powers. For if a more exact value of v' were obtained, the next term would contain y^4.

The point to which the rays converge when y is made indefinitely small is the focus conjugate to P; and we see, as before, that it is given by—

$$v = \mu u.$$

The aberration of the ray Rq is—

$$v' - v = \frac{\mu^2 - 1}{\mu} \cdot \frac{y^2}{2u}.$$

The aberration is from or towards the surface according as μ is $>$ or $<$ 1.

This investigation and those which follow show more clearly how, as the ray approaches nearer to the axis, the point at which it intersects the axis after deviation approaches to a definite limiting position.

To find where a Ray reflected at a Spherical Surface, and close to the Axis of the Surface, meets the Surface

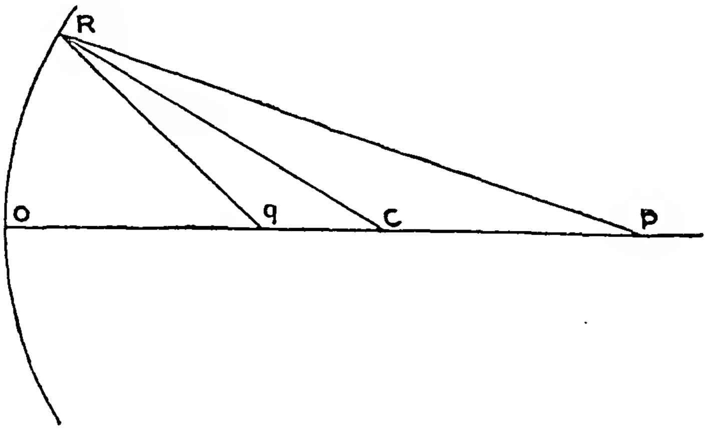


FIG. 73.

after Reflexion.—Let the ray P R after reflexion at R intersect the axis in q. Let CO = R. Let OP = u; Oq = v'. Let the perpendicular from R on OC = y.

Since RC bisects the angle PRq—

$$PR : Rq = PC : Cq;$$
$$\therefore PR \,.\, Cq = Rq \,.\, PC.$$

Now $PR^2 = PC^2 + CR^2 + 2PC \,.\, CR \cos RCO$;

And $\sin RCO = \dfrac{y}{R}$.

$$\therefore \cos RCO = \sqrt{1 - \frac{y^2}{R^2}}$$

$$= 1 - \frac{y^2}{2R^2}, \text{ approximately.}$$

$$\therefore PR^2 = (u - R)^2 + R^2 + 2(u - R)R\left(1 - \frac{y^2}{2R^2}\right)$$

$$= u^2 - \frac{u - R}{R} \cdot y^2$$

$$= u\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{u}\right\};$$

$$\therefore PR = u\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\}, \text{ approximately.}$$

Similarly—

$$Rq = v'\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

Thus we get—

$$(R - v')u\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\} = (u - R)v'\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

Divide by $uv'R$, and we get—

$$\left(\frac{1}{v'} - \frac{1}{R}\right)\left\{1 - \left(\frac{1}{R} - \frac{1}{u}\right)\frac{y^2}{2u}\right\} = \left(\frac{1}{R} - \frac{1}{u}\right)\left\{1 - \left(\frac{1}{R} - \frac{1}{v'}\right)\frac{y^2}{2v'}\right\};$$

$$\therefore \frac{1}{v'} + \frac{1}{u} = \frac{2}{R} + \left(\frac{1}{R} - \frac{1}{u}\right)\left(\frac{1}{v'} - \frac{1}{R}\right)\left(\frac{1}{v'} + \frac{1}{u}\right)\frac{y^2}{2}.$$

Substituting in the last term the approximate value of $\dfrac{1}{v'}$, got by neglecting this term, we get—

$$\frac{1}{v'} + \frac{1}{u} = \frac{2}{R} + \left(\frac{1}{R} - \frac{1}{u}\right)^2 \cdot \frac{y^2}{R}.$$

If v is the distance from O of Q, the focus conjugate to P—

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{R};$$

$$\therefore \frac{1}{v'} - \frac{1}{v} = \left(\frac{1}{R} - \frac{1}{u}\right)^2 \cdot \frac{y^2}{R}.$$

EXAMPLES. 57.

Find the L. C. M. of

1. 12 and 32. **2.** 76 and 98. **3.** 81 and 99. **4.** 320, 704.
5. 117, 192. **6.** 1224, 1696. **7.** 224, 336.
8. 754, 806. **9.** 957, 1001. **10.** 845, 899.
11. 779, 1197. **12.** 1287, 6281. **13.** 76, 95, 106.
14. 629, 851, 253. **15.** 265, 385, 495. **16.** 300, 906, 708.

17. Resolve 210 and 385 into their prime factors, and hence obtain their L. C. M.

18. Find the L. C. M. of 44, 54 and 72 by resolving them into their prime factors.

19. Find the L. C. M. of R3. 9*a*. 4*p*. and R7. 10*a*. 3*p*.

20. The H. C. F. and L. C. M. of two numbers are 16 and 192 respectively; one of the numbers is 48: find the other.

21. The H. C. F. and L. C. M. of two numbers are 10 and 30030 respectively; one of the numbers is 770: what is the other?

100. The following rule gives the most convenient method of finding the L. C. M. of several small numbers:

Place the numbers side by side in a line; divide by any one of the prime numbers 2, 3, 5, 7, 11,......which will divide any two at least of the given numbers exactly; set down the quotients thus obtained and the undivided numbers side by side; and proceed in this way until you get a line of numbers which are prime to one another. The continued product of all the divisors and the numbers in the last line will be the L. C. M. required.

Example 1. Find the L. C. M. of 12, 18, 20 and 105.

Process:

$$\begin{array}{r|l} 2 & 12,\ 18,\ 20,\ 105 \\ \hline 2 & 6,\ 9,\ 10,\ 105 \\ \hline 3 & 3,\ 9,\ 5,\ 105 \\ \hline 5 & 1,\ 3,\ 5,\ 35 \\ \hline & 1,\ 3,\ 1,\ 7 \end{array}$$

$\therefore$ L. C. M. $=2\times2\times3\times5\times3\times7=1260$.

Note. Work may be shortened by rejecting, at *any* stage, from the line any one of the numbers, which is a factor of any other number in the same line.

Thus, if it is required to find the L. C. M. of 6, 12, 15, 30 and 40, it will be sufficient to find the L. C. M. of 12, 30 and 40.

Dividing by $uv'R$, we get—

$$\mu\left(\frac{1}{R}-\frac{1}{v'}\right)\left\{1-\left(\frac{1}{R}-\frac{1}{u}\right)\frac{y^2}{2u}\right\}=\left(\frac{1}{R}-\frac{1}{u}\right)\left\{1-\left(\frac{1}{R}-\frac{1}{v'}\right)\frac{y^2}{2v'}\right\}.$$

$$\therefore \frac{\mu}{v'}-\frac{1}{u}=\frac{\mu-1}{R}+\left(\frac{1}{R}-\frac{1}{v'}\right)\left(\frac{1}{R}-\frac{1}{u}\right)\left(\frac{1}{v'}-\frac{\mu}{u}\right)\frac{y^2}{2}.$$

Substitute for v' in the small term containing y^2 from the approximate equation—

$$\frac{\mu}{v'}-\frac{1}{u}=\frac{\mu-1}{R}.$$

Thus we get—

$$\frac{\mu}{v'}-\frac{1}{u}=\frac{\mu-1}{R}+\frac{1}{\mu^2}\left(\frac{1}{R}-\frac{1}{u}\right)^2\left(\frac{1}{u}+\frac{\mu-1}{R}-\frac{\mu^2}{u}\right)\frac{y^2}{2}$$

$$=\frac{\mu-1}{R}+\frac{\mu-1}{\mu^2}\left(\frac{1}{R}-\frac{1}{u}\right)^2\left(\frac{1}{R}-\frac{\mu+1}{u}\right)\frac{y^2}{2}.$$

To find the Aberration of a Ray passing with Small Excentricity through a Thin Lens.—Let P B A be the axis of the lens; R and S the radii of its front and back surfaces.

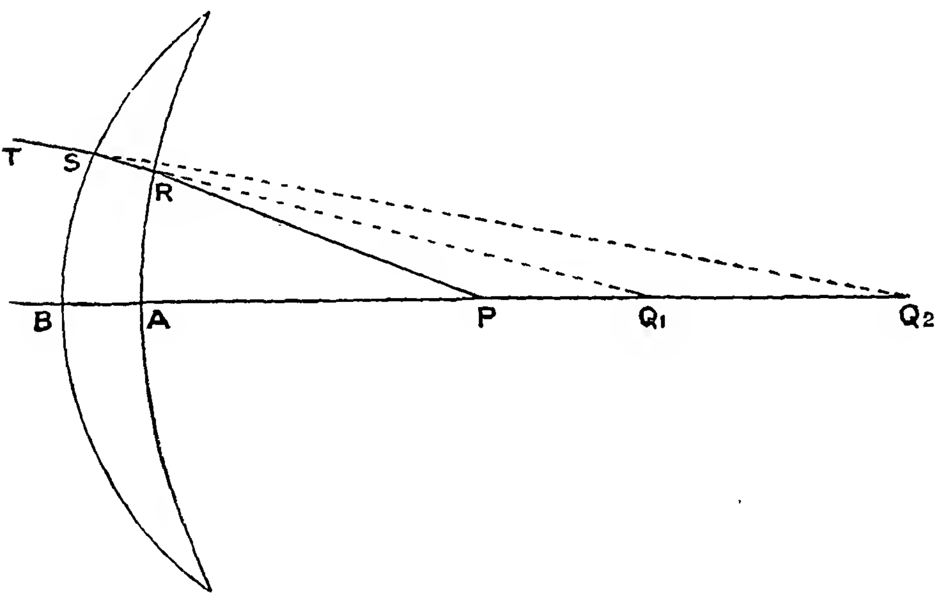


FIG. 75.

Let P R S T be the course of the ray. Let S R and T S produced meet the axis in Q_1, Q_2, at distances v_1, v_2 from the lens. Let A R, and therefore B S, nearly $= y$.

Then we have—

$$\frac{\mu}{v_1}-\frac{1}{u}=\frac{\mu-1}{R}+\frac{\mu-1}{\mu^2}\left(\frac{1}{R}-\frac{1}{u}\right)^2\left(\frac{1}{R}-\frac{\mu+1}{u}\right)\frac{y^2}{2}.$$

This equation connects u, v_1, and R. To find a similar equation connecting v_2, v_1, and S, notice that a ray along T S proceeding to Q_2, would, on refraction at the surface R S, proceed to Q_1. Thus in the above we have to replace u, v_1, and R by $-v_2$, $-v_1$, and $-S$; or we may change all signs and write v_2, v_1, and S. Thus we get—

$$\frac{\mu}{v_1}-\frac{1}{v_2}=\frac{\mu-1}{S}+\frac{\mu-1}{\mu^2}\left(\frac{1}{S}-\frac{1}{v_2}\right)^2\left(\frac{1}{S}-\frac{\mu+1}{v_2}\right)\frac{y^2}{2}.$$

Subtracting, we get—

$$\frac{1}{v_2}-\frac{1}{u}=(\mu-1)\left(\frac{1}{R}-\frac{1}{S}\right)+\frac{\mu-1}{\mu^2}\left\{\left(\frac{1}{R}-\frac{1}{u}\right)^2\left(\frac{1}{R}-\frac{\mu+1}{u}\right)\right.$$
$$\left.-\left(\frac{1}{S}-\frac{1}{v_2}\right)^2\left(\frac{1}{S}-\frac{\mu+1}{v_2}\right)\right\}\frac{y^2}{2}.$$

If v is the limiting value of v_2, we have—

$$\frac{1}{v}-\frac{1}{u}=(\mu-1)\left(\frac{1}{R}-\frac{1}{S}\right).$$

We may use the approximate value of v_2 got from this in the term containing y^2; and write—

$$\frac{1}{v_2}-\frac{1}{u}=(\mu-1)\left(\frac{1}{R}-\frac{1}{S}\right)+\frac{\mu-1}{\mu^2}\left\{\left(\frac{1}{R}-\frac{1}{u}\right)^2\left(\frac{1}{R}-\frac{\mu+1}{u}\right)\right.$$
$$\left.-\left(\frac{1}{S}-\frac{1}{v}\right)^2\left(\frac{1}{S}-\frac{\mu+1}{v}\right)\right\}\frac{y^2}{2}.$$

The aberration of the ray ST is—

$$v_2-v=-\frac{\mu-1}{\mu^2}\left\{\left(\frac{1}{R}-\frac{1}{u}\right)^2\left(\frac{1}{R}-\frac{\mu+1}{u}\right)\right.$$
$$\left.-\left(\frac{1}{S}-\frac{1}{v}\right)^2\left(\frac{1}{S}-\frac{\mu+1}{v}\right)\right\}\frac{v^2y^2}{2}.$$

The aberration for a ray parallel to the axis is got by putting $u=\infty$; $v=f$; and is—

$$-\frac{\mu-1}{\mu^2}\left\{\frac{1}{R^3}-\left(\frac{1}{S}-\frac{1}{f}\right)^2\left(\frac{1}{S}-\frac{\mu+1}{f}\right)\right\}\frac{f^2y^2}{2}.$$

R and S can be so chosen, subject to the condition that the lens shall have a given focal length, as to make the numerical value of this aberration a minimum.

If $\mu=\frac{3}{2}$, as is approximately the case for crown glass, the lens which will produce least aberration in a parallel pencil along the axis is such that $S=-6R$. Such a lens is called a *crossed lens.*

It would be possible so to choose the values of R and S for a lens of given focal length as to make the above expression, for the aberration of a ray from any point on the axis, vanish for certain values of u. The corresponding object-points are called **aplanatic foci** of the lens. Such a lens, it would be found, must have one surface convex and the other concave. An ordinary converging lens, with both surfaces convex, or with one plane, always produces a positive aberration; and a diverging lens, with both surfaces concave, or with one plane, always produces a negative aberration. It will be useful to notice that what these conclusions come to is this: any ordinary lens will produce aberration in such a manner that its marginal portions will behave like a lens of numerically shorter focal length than its central portion.

Suppose we have a pencil R S q reflected or refracted at a surface, so that Q q is its aberration. Take two rays, T q', U q',

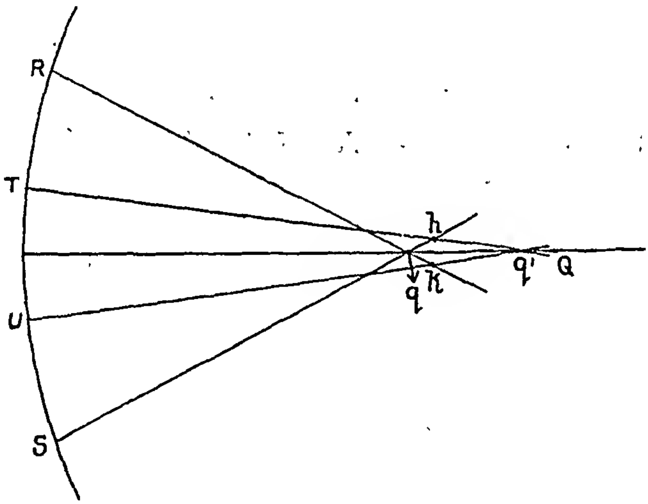


FIG. 76.

in the plane with R q, S q intersecting the axis in q', and meeting these rays in $h\,k$. Then $h\,k$ disappears when the rays T q', U q' lie on the axis, or when they coincide with R q, S q. So that for some intermediate position of T q', U q', $h\,k$ is a maximum. Let the figure represent this position. Then a circle on $h\,k$ as diameter, and at right angles to the axis, is the least area through which the entire pencil passes. This circle is called the **least circle of aberration.**

Suppose we take a series of normal sections to a very narrow astigmatic pencil, between its focal lines, the direction of these being at right angles to each other. As the section passes from the first to the second focal line, its breadth in the

direction of the first focal line diminishes to zero, and its breadth in the direction of the second focal line increases from zero. Thus in some position this section will have equal breadths in the directions of the two focal lines. This section is called **the circle of least confusion** of the pencil.

When an image of an object is seen by astigmatic pencils, we may regard the image as being the assemblage of all the circles of least confusion of the pencils.

The position and diameter of the circle of least confusion may be easily found. Let the orthogonal section of the pencil

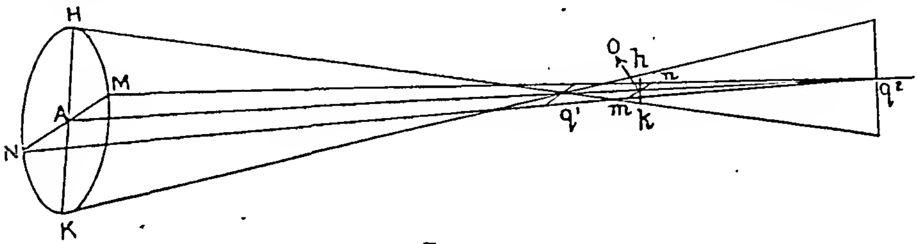


FIG. 77.

at A have breadths in the primary and secondary plane HK = a; MN = b. Let $Aq_1 = v_1$; $Aq_2 = v_2$. Let O be the centre of the circle of least confusion $h\,m\,k\,n$. And let AO = x. Then—

$$\frac{hk}{a} = \frac{x - v_1}{v_1}$$

$$\frac{mn}{b} = \frac{v_2 - x}{v_2}.$$

$$\text{And } hk = mn,$$

$$\therefore \frac{b}{a} = \frac{v_2(x - v_1)}{v_1(v_2 - x)},$$

$$\therefore x(av_2 + bv_1) = (a + b)v_1v_2,$$

$$\therefore x = \frac{(a + b)v_1v_2}{av_2 + bv_1}.$$

Again—

$$hk = \frac{a(x - v_1)}{v_1}$$

$$= \frac{ab(v_2 - v_1)}{av_2 + bv_1}.$$

Let us consider a broad pencil of light undergoing reflexion or refraction at a spherical surface. Consider the reflected or refracted rays which lie close to one plane through the axis—the plane of the paper. These rays do not meet in

one point in the plane. But any assemblage of them making up a small astigmatic pencil, such as R S, will converge to a primary focal line, q_1, at right angles to the plane. Thus q_1 is the region of maximum concentration of the light in the pencil R S. The various little astigmatic pencils made up of the rays we are considering give rise to a series of primary focal lines, q_1, q_1', q_1'', etc., arranged along a curve in the plane. This curve is a region of maximum illumination or of maximum concentration of the rays of the reflected or refracted pencil; it is the locus of the intersections of the successive rays in the plane. The curve is called a **caustic curve.**

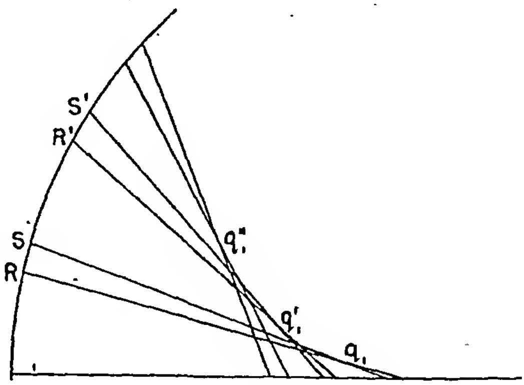


FIG. 78.

Considering the whole of the reflected or refracted pencil, we see that there is a surface which is a region of maximum illumination. This surface may be got by rotating the caustic curve about the axis. It is called a **caustic surface.**

A caustic by reflexion may be shown by placing a strip of bright metal bent into the form of a circular arc on a sheet of white paper, and so that a strong light, such as the light from the sun, may fall on the inner surface. The reflected light will give a caustic on the sheet of paper.

Another example of the caustic of a circle is the bright curve that may be seen on the top of a cup of tea by reflexion of light from the inside of the cup.

Caustic formed by Parallel Rays reflected at Cylindrical Mirror.—Suppose parallel light falls on a concave mirror in the form of a portion of a cylinder, the light being at right angles to the axis of the cylinder.

Let any ray, Q R, meet the surface at R (Fig. 79). Join R to O, the centre of the circular section of the cylinder through Q R. Bisect O R at T. Describe a circle with O T as radius, and another on T R as diameter, having centre O'. Let the latter circle be met by the reflected ray from R in P; and draw O F parallel to Q R. Then—

$$\angle PO'T = 2\angle PRO' = 2\angle TOF.$$

$$\text{And } O'T = \tfrac{1}{2}OT;$$

$$\therefore \text{arc } TP = \text{arc } TF.$$

Thus if the circle R P T rolls on the circle T F, the point of R P T which is in one position of it in coincidence with F will trace out a curve on which P is a point. The line R P touches

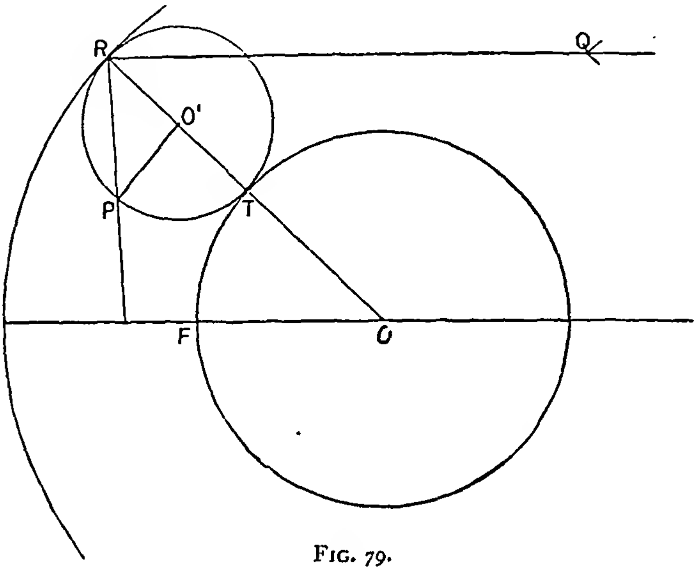


FIG. 79.

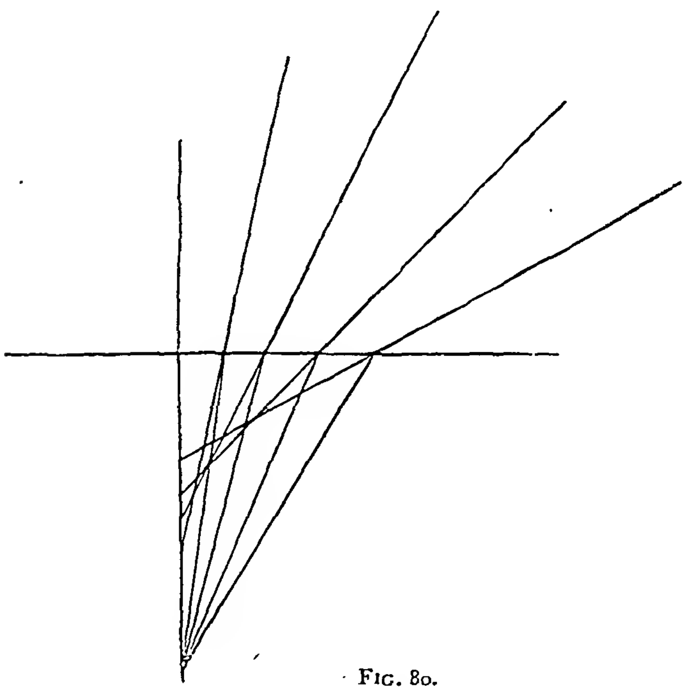


FIG. 80.

this curve at P. For at any instant the circle R P T is turning about its point of contact T with T F, and thus the direction of

motion of P, or of the curve through P, is at right angles to T P, that is, along P R. This curve, therefore, is touched by all the reflected rays such as R P It is therefore the caustic formed by these rays. The curve is an *epicycloid*.

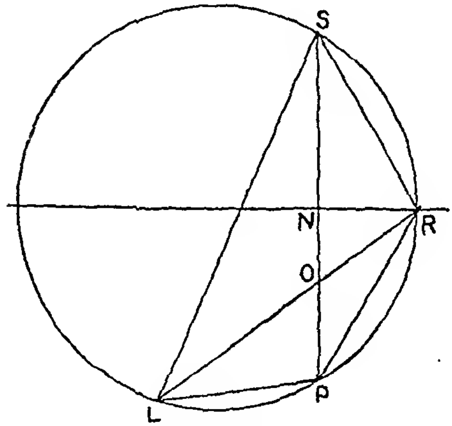


FIG. 81.

Caustic, formed by Refraction at a Surface, of Rays coming from a Point ($\mu < 1$).—We have in this case a virtual caustic. We can find the geometrical form of it by the following device.

Let P R be an incident ray. Draw P N normal to the surface, and produce it, making NS = PN. Let the refracted ray produced backward meet P S in O, and the circle, through P, R, S, in L.

Then, since LO bisects the angle PLS—

$$\frac{LS}{LP} = \frac{SO}{OP};$$

$$\frac{SL + LP}{LP} = \frac{SP}{OP};$$

$$SL + LP = SP \cdot \frac{LP}{OP}$$

$$= SP \cdot \frac{\sin LOP}{\sin OLP}.$$

Now, LOP is equal to the angle of refraction at R; and OLP = RSP = RPS = angle of incidence.

$$\therefore SL + LP = \frac{2PN}{\mu}$$

Thus the locus of L is an ellipse; and the refracted ray is always a normal to this ellipse. The caustic is touched by all the normals to the ellipse; that is, it is the *evolute* of the ellipse.

CHAPTER V.

COMPOSITE CHARACTER OF LIGHT.

Newton's Experiment.—Newton showed, by the following experiment, that the light from the sun, or, as a rule, from any source of illumination, is not simple in character, but composite, consisting of various sorts of simple light, which differ from each other in colour and in their degrees of refrangibility by a given refracting substance.

Sunlight is allowed to enter a darkened room through a small hole, O, in a shutter, and to fall on a white screen.

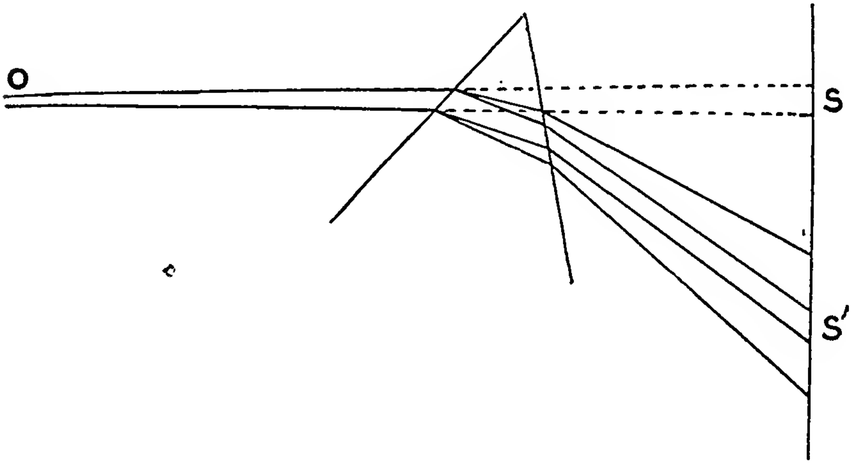


FIG. 82.

The light may be direct from the sun, or reflected by a mirror. An image of the sun would be formed on the screen at S, as in a camera obscura. Now a prism is interposed in the path of the rays, so as to deviate them and cause the image to be formed at another part of the screen, at S′.

It is noticed about this image—

(1) That it is considerably elongated in the direction at right angles to the edge of the prism.

(2) That it is brilliantly coloured with all the colours of the rainbow, from red in the least deviated portion to violet in the most deviated.

The colours through which the image passes from red to violet are innumerable in variety of tints. Newton considered them to be seven in number, namely, *red*, *orange*, *yellow*, *green*, *blue*, *indigo*, *violet*.

The variegated image thus produced by the sun's light is called a *solar spectrum.*

If a coloured screen is used to receive the image, the spectrum will not be complete, some of the colours appearing in their proper places, that is, where they would be on a white screen, but the rest of the screen being unilluminated, or, at the most, comparatively dark. The screen will be illuminated in that portion only of the spectrum which corresponds to its own colour. Thus a screen of pure red will only show illumination in the red part of the spectrum. The colour of a screen may, however, not be the same as any one of the colours of the spectrum. In that case it is made up of more than one, and it will show illumination in more than one part of the spectrum, or, in an extended part. This is the case with the great majority of substances, there being few substances in nature whose colours are simple.

A substance will, then, be illuminated by light of its own colour only; or of such colours as its colour contains, if its colour is composite. And we infer that a white screen contains all the colours in the solar spectrum, that is, that white is made up of all these colours. Further, since these are all the simple colours that can be obtained from the sun's light, it follows that the sun's light is composed of light of all the colours that go to make up white, and in the right proportions, or is *white light.*

In the spectrum formed as described above, since the light passes by imperceptible variations through the various degrees of refrangibility, and since the light of any definite colour would form an image of finite dimensions, it follows that the images of various colours encroach upon or overlap each other. Thus the illumination at any point of the spectrum is not produced by light of one simple kind, but by lights of all degrees of refrangibility between certain limits. The spectrum so produced is said to be an impure one.

Pure Spectrum.—A pure spectrum is one in which the light at each point is simple or of a definite refrangibility. To obtain a pure spectrum, the aperture used to admit the light must be a very narrow slit, A (Fig. 83); there is generally some arrangement for adjusting the width of it. A lens, L, is used to produce an image, B, of the slit on a screen. A prism, P, is placed between the lens and the screen, with its edge parallel to the slit, and rotated till the deviated image of A is in the position of minimum deviation. Since there is not very much difference in the deviations produced in the images formed by the various

kinds of light, they will all be practically in their positions of minimum deviation for the same position of the prism. The prism now forms at C, with the light of any simple kind, a real image of the slit, at the same distance from itself as B was.

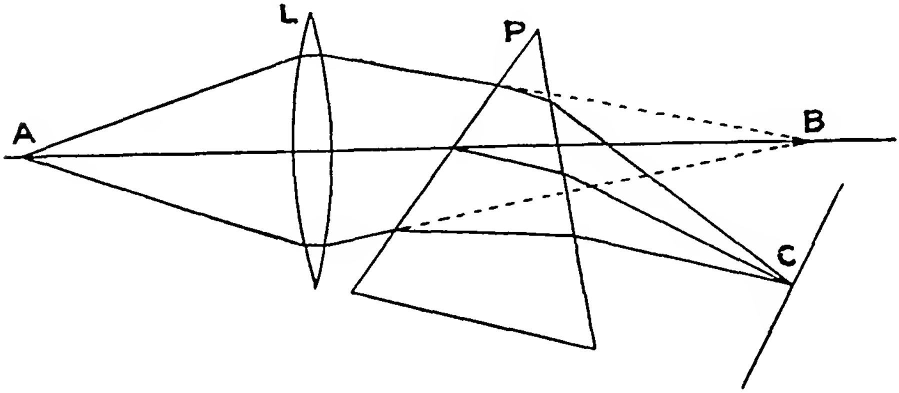


FIG. 83.

The screen is then placed to receive the spectrum produced at C, the assemblage of all the deviated images of the slit, its distance from the prism being made the same as the distance of B from the prism.

On account of the finite width of the slit, there will still be more or less overlapping of the images at C. But the slit can be made extremely narrow, so that the spectrum obtained will be, for all practical purposes, a pure one.

The light at any point of the pure spectrum, being quite simple or of a definite refrangibility, is called *homogeneous*, or *monochromatic.* The composition of light cannot be inferred from its colour, for two lights precisely alike in appearance may each be composed of monochromatic constituents such that those in the one light are entirely different from those in the other; and a monochromatic colour may be exactly matched by a combination of other colours all quite different from it.

A pure spectrum may also be obtained by letting the light from the slit fall first on the prism set in the position of minimum deviation, and then letting it fall on a lens which will produce an image of the slit formed by the light of each separate colour; a screen being set to receive these images which constitute the spectrum. The diagram (Fig. 84) shows this arrangement.

The simplest way, however, to see a pure spectrum is to let the eye itself bring to a focus the rays of each colour proceeding from the slit and refracted through the prism.

That is, in the above diagram, the lens of the eye takes the place of the lens, and the retina takes the place of the screen. Let the slit be at the distance of most distinct vision from the

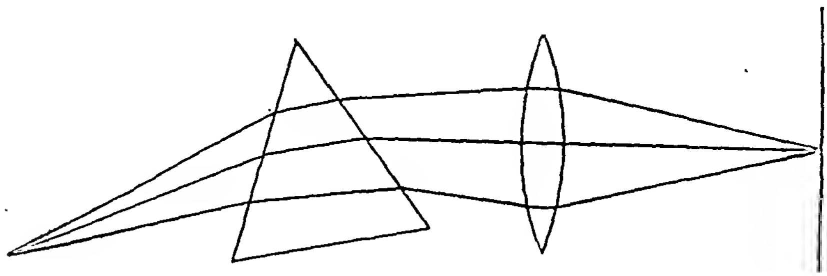


FIG. 84.

eye, the prism placed before the eye with its edge parallel to the slit. Turn the prism till the image, or series of coloured images, of the slit is at the position of minimum deviation. These images are now also at the distance of most distinct vision. The eye, being focussed on them, will see a pure spectrum.

When a composite ray of light is refracted, the constituent rays are deviated by different amounts. This is called **dispersion** of the light; and the angle between two given rays after deviation is called **the dispersion of the two rays.**

Let μ_r, μ_v, μ be the indices of refraction of a given medium for the extreme red and violet rays, and for the rays of mean refrangibility. Let D_r, D_v, D be the minimum deviations which would be produced in these rays by a prism of the substance of very small refracting angle, i. Then—

$$D_r = (\mu_r - 1)i,$$
$$D_v = (\mu_v - 1)i,$$
$$D = (\mu - 1)i.$$

Then the limiting ratio of dispersion to deviation is $\dfrac{D_r - D_v}{D}$, and this is equal to $\dfrac{\mu_r - \mu_v}{\mu - 1}$.

This fraction is called the **dispersive power** of the substance.

Recomposition of White Light.—If the colours into which white light has been broken up by dispersion be recombined, white light will again be produced. This can be done in several ways.

1. A prism exactly like that used to produce dispersion may be placed to receive the light just as it leaves the first, having its faces parallel to those of the first, and its refracting edge

turned the other way. Then every ray will be deviated by this prism just as much as it was by the first, and in the opposite sense. Thus any composite ray is dispersed by the first prism, and the rays into which it is broken up are brought into parallelism by the second. Suppose a pencil, A B C D, to pass through the two prisms, and the emerging light to fall on a screen at *a b c d*. Take the two extremes,

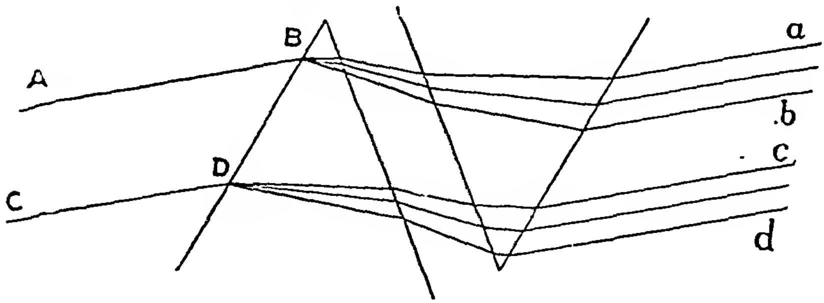


FIG. 85.

A B, C D, of the pencil. The ray A B gives rise to a series of rays falling on the screen from *a* to *b*, and giving the light which varies from red to violet as we pass from *a* to *b*. The ray C D gives rise, in the same way, to light on the screen varying from red to violet as we pass from *c* to *d*. And similarly for intermediate rays. All the portion from *b* to *c* will be illuminated with rays of all colours, and will appear white. In the boundary portion *a b*, the colours of the spectrum successively disappear as we pass from *b* to *a*, the violet being first wanting just beyond *b*, then as we go on towards *a* the blue *as well as* the violet, and so on, till at *a* the extreme red only remains. This edge of the illuminated area is thus coloured with a reddish tinge. In the same way in the other edge, *c d*, the colours of the spectrum successively disappear as we pass from *c* to *d*, the red going first. The edge is coloured a bluish-green.

2. If the light from the dispersing prism is received on a convex lens or a concave mirror, and then brought to a focus on a screen placed so that it and the prism are at conjugate foci, the colours will be recombined on the screen, and give rise to white light, at any rate in the middle of the portion they illuminate.

3. If a cardboard disc be coloured in sectors with the colours of the spectrum, and then spun rapidly about its centre, the appearance will be whitish, on account of the persistence of the visual impressions produced by the colours.

Dispersion produced by Lens.—Suppose a small object to be illuminated by white light; and let a lens be used to form an image of the object. If the light proceeding from the object were of one simple colour, the substance of the lens would have a definite refractive index for it, and the image would be formed in a definite position. But the indices of refraction are different for the different colours; the rays of various colours will be variously deviated; and a series of images, of various colours, will be formed, at slightly different distances from the lens.

This property of the lens, whereby the rays do not converge to a definite focus, but are spread out according to their colours, even when only a small portion of the lens round the axis is used, is called **chromatic aberration.**

The diagram shows the way in which the rays of various colours, which have fallen on a lens from a luminous or visible

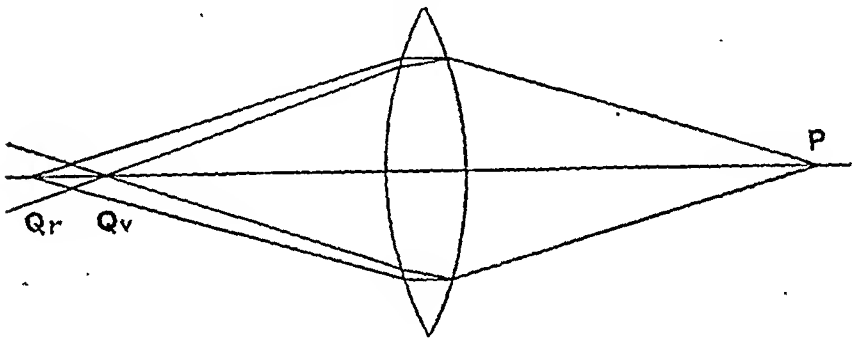


FIG. 86.

point, P, proceed on leaving the lens, giving rise to a series of coloured conjugate foci, or images of P, ranged along the axis from Q_r the red, to Q_v the violet, image; Q_v being nearest to the lens for the case drawn, because the violet rays are most refrangible.

This effect may be shown in the following way: Hold a white screen so as to receive the image of P. When the screen is held at Q_v, an image of P will be formed on it of the violet rays. The other sets of rays will not yet have come to a focus, but will be spread out, and especially the red rays. The image formed of P will have an edge that is badly defined, and coloured with a reddish tinge. When the screen is held at Q_r, an image is formed of the red rays; the other sets of rays have all passed their foci, and will be spreading out, especially the violet. The image will have a badly defined edge coloured with a bluish-green tinge. The best image will

be formed somewhere between Q_r and Q_v, where there is least chromatic aberration of the rays.

Chromatic aberration is a serious difficulty in the use of lenses, and we must consider how this difficulty has to be overcome. Chromatic aberration is produced in a lens in this way: An excentrical ray which falls on the lens passes into and out of the lens at points where the surfaces are not parallel. The ray is thus deviated by the portion of the lens through which it passes, in just the same way as it would be deviated by a prism. And it undergoes dispersion just as it would on passing through a prism. If the rays could be deviated without dispersion, that is, so as all to be brought to the same focus, there would be no chromatic aberration. The problem, then, is to produce deviation by refraction without dispersion. It will be simpler, first, to consider how this may be done with prisms.

Suppose a prism of a given sort of glass to produce dispersion in a parallel pencil of light, the prism being set to produce minimum deviation. Let the light, after passing through this prism, fall on a second one set with its edge parallel to that of the first, and turned to produce deviation in the opposite direction, and so as also to produce minimum deviation of the light which falls upon it. Let the second prism be of the same material as the first. Then if it is made so as just to cut out the dispersion produced by the first, or to bring the rays into parallelism again, it must have the same refracting angle as the first. And, having the same refracting angle, it will also just cut out the deviation produced by the first prism. In this way, then, we can only destroy the dispersion at the expense of all the deviation. It remains to be seen whether with prisms of different substances we can do otherwise.

Newton inferred, from the limited number of experiments which he made, that it was impossible to combine two prisms to produce deviation without dispersion; that is, that whenever a beam of light passed through two prisms so as to emerge colourless, it also emerged parallel to its old direction. This would mean, for prisms of small angles, that the dispersion is always proportional to the deviation, whatever the substance of the prism; or that the dispersive powers of all substances are the same. We know now that this conclusion is not correct for all substances. If, for example, we use a crown-glass prism and a flint-glass prism, and they are made to produce equal deviations, the flint will produce more dispersion

for two given rays; or, if they are made to produce equal dispersions, the crown will produce more deviation. In flint glass, in fact, the ratio of dispersion to deviation is greater than in crown, or flint glass has a greater dispersive power than crown. A prism of crown and a prism of flint glass could, then, be made and arranged so that the flint would cut out all the dispersion produced by the crown, but leave in some of the deviation. In this way is solved the problem of producing deviation in a pencil of light, by means of prisms, without producing dispersion.

In what has been said we must, in strictness, be supposed to refer to the dispersion of *two* rays, say the extreme red and violet. For if these two rays be brought together in such a combination of prisms as that just described, it does not follow that the other rays will be combined with them. The dispersions between the various pairs of rays do not bear quite the same ratios in the two substances. Thus when the red and the violet are brought together, another ray, say the mean yellow, will not be brought quite into coincidence with them. The prisms producing the same dispersion between red and violet do not produce quite the same dispersion between red and yellow. Supposing, then, that the light came from a slit parallel to the edges of the prisms, there will be formed on the screen an image of the slit which is slightly coloured, the colours on one side consisting of a combination of colours from the two ends of the spectrum, and those on the other of a combination of colours from the middle. The appearance thus produced on the screen is called a **secondary spectrum.** This is much less coloured and spread out than the primary spectrum.

The fact which gives rise to secondary spectra, that the dispersions between the various pairs of rays in different refracting substances are not proportional, is called **irrationality of dispersion.**

We may use prisms of three different substances, so as to recombine three different rays of the spectrum, say red, yellow, and violet. The appearance then produced on the screen would present very little colour indeed. It would be a **tertiary spectrum.**

With a combination of prisms, say of two, we may use a very narrow slit and a lens to get the spectrum produced by white light on a white screen, just as was done with a single prism. We should then have resulting a pure secondary spectrum. We should have the images formed by red and

violet lights superposed (these being the colours brought together); and side by side with these the images resulting from other combinations of pairs of colours, the whole being spread out only a very little way over the screen. As the slit is made broader, the image will become white in the middle, the edges only being coloured, one of them with colours from the two ends of the primary spectrum, the other with colours from the middle.

A combination of prisms or lenses which produces a colourless image is said to be an **achromatic combination.**

We have next to consider the question of making a combination of lenses which shall be achromatic. Now, such a combination may be required to be achromatic in one of two distinct ways. We shall, in all cases, begin by supposing that the lenses have a common axis.

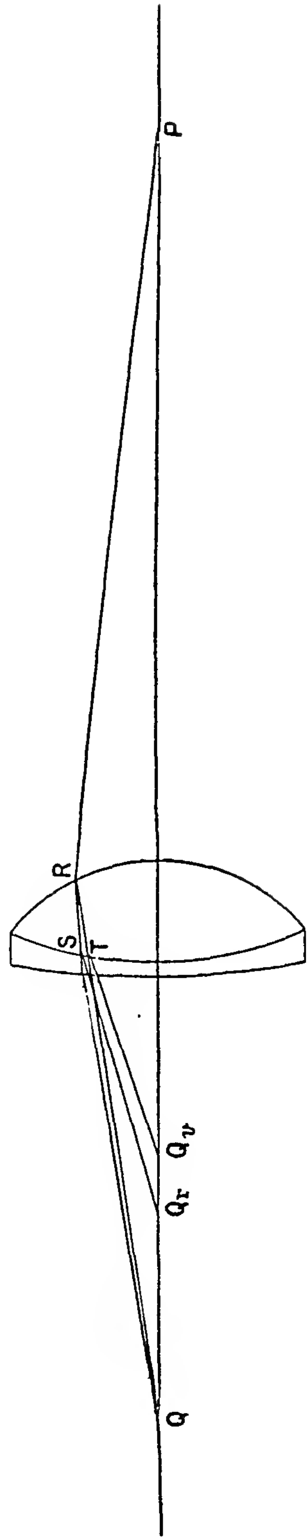
1. The combination may be made achromatic for images formed by *centrical* pencils. Then the images of a visible point, formed by such small pencils of the various colours, must all be superposed at another point. If the lenses are separated by appreciable distances, that the pencils may be centrical for all, it is clear that point and image must be on the common axis. If the lenses are of inconsiderable thickness and in contact, the same oblique pencil can be approximately centrical for all. In this case points close to, but not on, the axis would also have formed of them achromatic images by means of centrical pencils. A combination of lenses to be used to form an image on a screen requires to be made achromatic in this sense; so also do the object-glasses of microscopes and of ordinary telescopes, for it is centrical pencils that are used in all these cases.

2. The combination may be made *achromatic for excentrical pencils*, that is, so that a ray of light passing excentrically through the lenses may have its various constituents emerging parallel, and therefore, practically, in coincidence. This is necessary for a combination of lenses which is to be looked through; for an eye sees the various points of an object through various parts of the lens, and thus by excentrical pencils. Compound eye-pieces of microscopes and telescopes have to be made achromatic in this sense.

We shall now consider how a lens may be made which will form a real image of an object, and bring together the red and the violet rays by which it is formed.

Suppose we have a point, P, illuminated by white light, on the axis of a convex lens of crown glass. The rays from P along

PR will undergo dispersion, the red being least deviated, and the violet most; so that the red ray will take the course PRSQ_r, and the violet the course PRTQ_v. Now, behind the crown lens let a concave one of flint glass be placed, having the same axis as the first. The rays SQ_r, TQ_v will be deviated away from the axis, and the latter will be the more deviated; so that they may, by a suitable combination of lenses, be brought to meet the axis in the same point Q. For a combination of two lenses to produce a focus which is achromatic for two colours, conjugate to a point P on the axis, its focal length must be the same for both colours, and thus it will produce achromatic conjugate foci for all points on the axis.



Now, suppose the focal length of the lenses so chosen as to produce an achromatic image, Q, of the point P on the axis. Then if P is taken at the same distance as before from the common centre, but now a little off the axis, its images Q_r, Q_v, formed by the first lens, will be at the same distances from the centre as before, for the distance of image and object are connected by the same formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

whether the line joining them is the axis or is slightly inclined to the axis. And these points, Q_r, Q_v, will give images on the line joining them to the centre, all together, and at the same distance as before. Thus in this case, too, the image of P is achromatic. And the combination will form an achromatic image of a small object on the axis.

It should be noticed that this combination, having a centre and a focus, may be treated as a single lens for purposes of graphic construction, determination of magnification, and so on.

Condition of Achromatism for a Combination of Two Thin Lenses for Pencils passing centrically and close to the Axis.—Suppose two lenses combined as just described, their thicknesses being inconsiderable. To be achromatic for two colours, their combined focal lengths for each of the colours must be the same. Let the two colours have refractive indices μ_1, μ_1' in the first lens; and μ_2, μ_2' in the second. Let the four radii of curvature of the lenses be r_1, s_1; r_2, s_2. Let the focal lengths of the first lens for the two colours be f_1, f_1'; and of the second f_2, f_2'. Let the focal lengths of the combination for the two colours be F, F'. Then—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$
$$= (\mu_1 - 1)\left(\frac{1}{r_1} - \frac{1}{s_1}\right) + (\mu_2 - 1)\left(\frac{1}{r_2} - \frac{1}{s_2}\right);$$
$$\frac{1}{F'} = \frac{1}{f_1'} + \frac{1}{f_2'}$$
$$= (\mu_1' - 1)\left(\frac{1}{r_1} - \frac{1}{s_1}\right) + (\mu_2' - 1)\left(\frac{1}{r_2} - \frac{1}{s_2}\right).$$

But F = F'.

$$\therefore 0 = (\mu_1' - \mu_1)\left(\frac{1}{r_1} - \frac{1}{s_1}\right) + (\mu_2' - \mu_2)\left(\frac{1}{r_2} - \frac{1}{s_2}\right);$$
$$\therefore \frac{\mu_1' - \mu_1}{\mu_1 - 1}\cdot\frac{1}{f_1} + \frac{\mu_2' - \mu_2}{\mu_2 - 1}\cdot\frac{1}{f_2} = 0.$$

Or, by the differential calculus, since focal length, F, of the combination is given by—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2},$$

the differential of this, in passing to another refractive index, must vanish.

$$\therefore d\cdot\frac{1}{f_1} + d\cdot\frac{1}{f_2} = 0.$$

$$\text{But } \frac{1}{f_1} = (\mu_1 - 1)\left(\frac{1}{r_1} - \frac{1}{s_1}\right),$$

$$\therefore d \cdot \frac{1}{f_1} = d\mu_1 \left(\frac{1}{r_1} - \frac{1}{s_1} \right) \doteq \frac{d\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1}.$$

$$\text{So } d \cdot \frac{1}{f_2} = \frac{d\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2}.$$

$$\therefore \frac{d\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1} + \frac{d\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2} = 0.$$

If we require to find what lenses will make a combination achromatic for two given colours, and having a given focal length, F, for them, this condition and the equation—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

are sufficient to give us f_1 and f_2.

The above condition of achromatism is that the sum of the products for the two lenses of dispersive powers and reciprocals of focal lengths shall be zero.

For a similar combination of three lenses to be achromatic for two given colours, we would get a similar condition in three terms. Having the three focal lengths at disposal, we can make them satisfy another condition, so that we can make the combination achromatic for one of the two given colours and a third. So that with three lenses of different substances we can make a combination achromatic for three colours.

In general, with any number of thin lenses combined as above, the condition for achromatism for two colours may be written—

$$\Sigma \left(\frac{d\mu}{\mu - 1} \cdot \frac{1}{f} \right) = 0.$$

Suppose there are n lenses; then, if we choose n colours by taking them two and two, we can get $n - 1$ *independent* equations, of which the above is a type. These, with the equation among the f's for the focal length of the combination, are n equations, from which to find the $n f$'s. Thus, with n lenses we can recombine n colours.

The number of independent conditions among the f's is limited as follows: Suppose there are, for example, three colours, red, yellow, and violet, to be combined. To combine the red with the violet introduces one condition, and to combine the red with the yellow another. To combine the yellow with the violet introduces no new condition; this is done by combining the red with each of them. In the same way, n colours will be combined by combining one with each

of the other $n - 1$; thus giving $n - 1$ independent conditions or equations.

Suppose we have two lenses of focal lengths f_1, f_2, and refractive indices μ_1, μ_2, for a given colour. Let them be separated by an interval a. To find the condition that they should be achromatic for a parallel pencil along the axis.

The distance from the second lens at which the light is brought to a focus is given by—

$$\frac{1}{v} = \frac{1}{f_1 + a} + \frac{1}{f_2}.$$

Now let us pass to another colour, so that the f's and the μ's undergo small changes, df_1, df_2, $d\mu_1$, $d\mu_2$. The corresponding variation of v must be zero.

We have already seen that—

$$d \cdot \frac{1}{f_1} = \frac{d\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1}.$$

We may write—

$$\frac{1}{v} = \frac{\frac{1}{f_1}}{1 - \frac{a}{f_1}} + \frac{1}{f_2}.$$

And since the differential of this must vanish, we must have, by elementary differential calculus—

$$0 = \frac{\left(1 + \frac{a}{f_1}\right) d \cdot \frac{1}{f_1} - \frac{1}{f_1} \cdot a \cdot d \cdot \frac{1}{f_1}}{\left(1 + \frac{a}{f_1}\right)^2} + d \cdot \frac{1}{f_2};$$

$$\frac{d \cdot \frac{1}{f_1}}{\left(1 + \frac{a}{f_1}\right)^2} + d \cdot \frac{1}{f_1} = 0;$$

$$\frac{\frac{d\mu_1}{\mu_1 - 1} \cdot \frac{1}{f_1}}{\left(1 + \frac{a}{f_1}\right)^2} + \frac{d\mu_2}{\mu_2 - 1} \cdot \frac{1}{f_2} = 0;$$

$$\frac{f_1}{(a + f_1)^2} \cdot \frac{d\mu_1}{\mu_1 - 1} + \frac{1}{f_2} \cdot \frac{d\mu_2}{\mu_2 - 1} = 0.$$

This result could be obtained by algebra, by supposing

μ_1, μ_2 to vary slightly and become μ_1', μ_2'; but the process would be much longer.

The Chromatic Aberration between the rays of two colours in a small pencil along the axis of a lens, after refraction through the lens, is the distance between the foci for these two colours.

The chromatic aberration of the pencil is the aberration between its extreme colours.

To find the chromatic aberration between two colours in a small axial pencil refracted through a lens.

Let u be the distance of the focus P of the pencil; v_r, v_v those of the conjugate foci for the red and violet rays; f_r, f_v the corresponding focal lengths of the lens; and μ_r, μ_v the refractive indices. The chromatic aberration is $\pm(v_r - v_v)$.

$$\text{Now } \frac{1}{v_r} - \frac{1}{u} = \frac{1}{f_r};$$

$$\frac{1}{v_v} - \frac{1}{u} = \frac{1}{f_v}.$$

Therefore, subtracting—

$$\frac{v_r - v_v}{v_r v_v} = \frac{1}{f_v} - \frac{1}{f_r}$$

$$= \frac{\mu_v - \mu_r}{\mu - 1} \cdot \frac{1}{f}.$$

f and μ being mean focal length and refractive index.

Now, v_r and v_v are very nearly equal, so that we may write $v_r v_v = v^2$, where v is the distance of the focus conjugate to P for the mean rays. Thus—

$$v_r - v_v = \frac{\mu_v - \mu_r}{\mu - 1} \cdot \frac{v^2}{f}.$$

Or, by the differential calculus—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f};$$

$$\therefore -\frac{dv}{v^2} = d \cdot \frac{1}{f};$$

$$dv = -\frac{d\mu}{\mu - 1} \cdot \frac{v^2}{f}.$$

in which dv and $d\mu$ stand for $v_v - v_r$ and $\mu_v - \mu_r$.

Suppose a combination has to be made **achromatic for excentrical pencils.** Then the rays of various colours from any point must enter the eye in coincidence. For this

condition to be fulfilled, the images, of various colours, of any small object formed by the last lens must have their corresponding points in straight lines with the eye. Now, supposing the pencils by which the eye sees the image to be inclined at small angles to the axis, and since the various coloured images are not far from each other, this condition is sufficiently attained by making these images all of the same size.

The combination of a thin convex and a thin concave lens in contact, which is achromatic for centrical pencils, is also achromatic for excentrical pencils. For the coloured images formed by both lenses of a small object are all in one place, and thus at the same distance from the common centre of the lenses. Thus they are all of the same size.

Consider two lenses, focal lengths for mean rays f_1, f_2, separated by a distance a. The condition that this combination should be achromatic for excentrical pencils coming from a small object on the axis will involve the position of the object. We shall consider merely the case in which the object is at a great distance; so that the combination may be achromatic for rays parallel to the axis; and we shall suppose the lenses made of the same material.

The first lens forms an image practically at its principal focus. This is at a distance, $a+f_1$, from the centre of the second lens. Suppose an image is formed in the second lens at a distance, v, from its centre. Then—

$$\frac{1}{v}=\frac{1}{f_2}+\frac{1}{a+f_1}.$$

Now, if b is the distance of the object from the first lens, the full magnification of the image is—

$$\frac{f_1}{b}\cdot\frac{v}{a+f_1}.$$

This must be constant for all the images.

$$\therefore \frac{1}{f_1}\cdot\frac{a+f_1}{v} \text{ must be constant.}$$

$$\text{Now } \frac{1}{f_1}\cdot\frac{a+f_1}{v}=\frac{1}{f_1}\left(1+\frac{a+f_1}{f_2}\right)$$
$$=\frac{1}{f_1}+\frac{1}{f_2}+\frac{a}{f_1 f_2}.$$

This must be constant for variations of f_1 and f_2 caused by variation in μ.

$$\therefore d \cdot \frac{1}{f_1} + d \cdot \frac{1}{f_2} + a\left(\frac{1}{f_1} \cdot d \cdot \frac{1}{f_2} + \frac{1}{f_2} \cdot d \cdot \frac{1}{f_2}\right) = 0.$$

Therefore, dropping the factor $\frac{d\mu}{\mu - 1}$ throughout, we get—

$$\frac{1}{f_1} + \frac{1}{f_2} + a\left(\frac{1}{f_1 f_2} + \frac{1}{f_2 f_1}\right) = 0;$$

$$a = -\frac{f_1 + f_2}{2};$$

which is the required condition.

EXAMPLE.

A compound achromatic lens of focal length 40 cms. is to be constructed of two thin crown-glass and flint-glass lenses in contact, the surfaces that are in contact having a common radius of 25 cms. The optical characters of the glasses employed being as follows, namely :—

	Dispersive power.	Refractive index for middle of spectrum.
Crown glass	0·21 ...	1·5
Flint glass	0·45 ...	1·6

calculate the radius of the second face of each lens, and establish the formulæ employed in the calculation. (Lond. B.Sc. Hons., 1884.)

CHAPTER VI.

DEFECTS IN IMAGES FORMED BY LENSES AND MIRRORS.

Defects in the Images formed by Lenses.—We have already considered the defects of colouring in the image formed by a lens, or chromatism; we shall here consider only the defects of form.

We have seen that a lens will only produce a true geometrical image of a point if the point is on its axis; but a very close approximation is produced, and generally sufficient for practical purposes, to a true geometrical image, of a small object through which the axis of the lens passes. We shall now consider a little more fully the shape of this image, and how far it resembles the object.

We have seen that a small object on the axis at a distance u gives rise to an image at a distance v; v being determined by the formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

Also that if the object be in one plane perpendicular to the axis, the image will, too, be in such a plane, and similar to the object; the ratio of the distance between two points of the image to that between the corresponding points of the object being $\frac{v}{u}$.

Now, suppose all the points of the object are not in the same plane perpendicular to the axis. Let the differences of

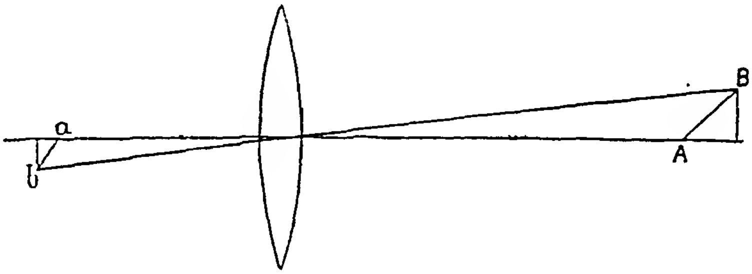


FIG. 88.

the u's for two points be du. This is very nearly the distance between the corresponding planes. Let the corresponding difference in the v's be dv.

Then since—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

we have—

$$\frac{dv}{v^2} - \frac{du}{u^2} = 0;$$

$$\therefore \frac{dv}{du} = \frac{v^2}{u^2}.$$

Thus the ratio of the size of the image to that of the object, when measured parallel to the axis, is not, as a rule, the same as when it is measured perpendicular to the axis.

The image is then *distorted* in this respect.

Curvature of Image.—Consider the image formed by centrical pencils by a lens, of a small object on the axis all in one plane perpendicular to the axis; say of a small straight line. We have seen that, to the second order of small quantities, the focal lines of any point of the object agree with each other and with the image, and this is given by the ordinary formula—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

To investigate the curvature of the image, however, we must go to a closer degree of approximation.

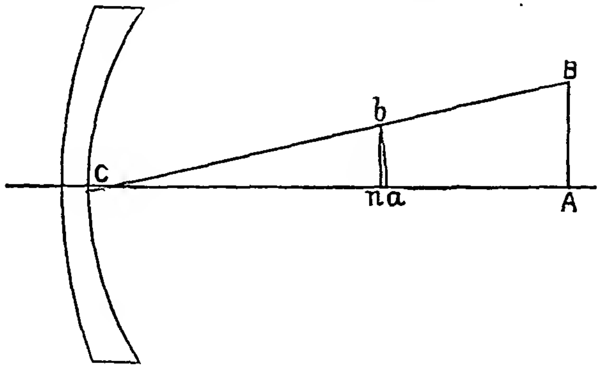


FIG. 89.

Suppose the small straight line A B (A being on the axis) to give the image *a b*. Let CB $= u$.

Now, B forms two focal lines in the neighbourhood of *b*, *b* being generally taken as the circle of least confusion between these two.

Let the distances of the focal lines from C be v_1, v_2.

Then we have—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{\mu \cos r - \cos i}{\cos^2 i}\left(\frac{1}{R} - \frac{1}{S}\right),$$

$$\frac{1}{v_2} - \frac{1}{u} = (\mu \cos r - \cos i)\left(\frac{1}{R} - \frac{1}{S}\right).$$

r and i being both small, we may write, correct to the *third* order of small quantities,—

$$\cos r = 1 - \frac{r^2}{2},$$

$$\cos i = 1 - \frac{i^2}{2},$$

$$(\cos i)^{-2} = 1 + i^2,$$

$$i = \mu r.$$

Thus—

$$\frac{\mu \cos r - \cos i}{\cos^2 i} = \left(\mu - \frac{\mu r^2}{2} - 1 + \frac{i^2}{2}\right)(1 + i^2)$$

$$= \left(\mu - 1 - \frac{i^2}{2\mu} + \frac{i^2}{2}\right)(1 + i^2)$$

$$= \mu - 1 + i^2\mu - \frac{i^2}{2} - \frac{i^2}{2\mu} \quad \text{(correct to}$$

the third order of small quantities)

$$= \mu - 1 + \frac{i^2}{2\mu}(2\mu^2 - \mu - 1)$$
$$= (\mu - 1)\{1 + \frac{i^2}{2\mu}(2\mu +)\}.$$

And—

$$\mu \cos r - \cos i = \mu - 1 - \frac{\mu r^2}{2} + \frac{i^2}{2}$$
$$= \mu - 1 - \frac{i^2}{2\mu} + \frac{i^2}{2}$$
$$= (\mu - 1)\left(1 + \frac{i^2}{2\mu}\right).$$

Thus—

$$\frac{1}{v_1} - \frac{1}{u} = \frac{1}{f}\{1 + \frac{i^2}{2\mu}(2\mu + 1)\},$$
$$\frac{1}{v_2} - \frac{1}{u} = \frac{1}{f}\left(1 + \frac{i^2}{2\mu}\right).$$

The distance, v, of the image b is, therefore, given by—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}(1 + ki^2),$$

where k is some positive constant, whether we take the circle of least confusion or one of the focal lines for the image.

Draw bn perpendicular to the axis. We have—

$$\frac{1}{Ca} - \frac{1}{CA} = \frac{1}{f}.$$

Multiply this equation by $\cos i$ (*i.e.* approximately by $1 - \frac{i^2}{2}$), and subtract from the above.

Then, since $u \cos i = CA$, we have—

$$\frac{1}{v} - \frac{\cos i}{Ca} = \frac{1}{f}\left(1 + ki^2 - 1 + \frac{i^2}{2}\right),$$
$$\frac{na}{v \cdot Ca} = \frac{(2k + 1)i^2}{2f}.$$

In the limit $i^2 v . Ca = bn^2$.

And if ρ is the radius of curvature of the curve ab at a, measured from a leftwards—

$$\frac{1}{2\rho} = \text{limit of } \frac{na}{bn^2};$$
$$\therefore \frac{1}{\rho} = \frac{2k + 1}{f}.$$

Example 4. Find the value of $\frac{311}{550}$ of ₨10. 2*a*. 6*p*.

Process : $\frac{311}{550}$ of ₨10 . 2 . 6 $=\frac{311}{550}$ of 1950*p*.

$=\frac{311\times1950}{550}p.=\frac{311\times39}{11}p.=\frac{12129}{11}p.$

$=1102\frac{7}{11}p.=91a.\ 10\frac{7}{11}p.=$ ₨5. 11*a*. $10\frac{7}{11}p$ *Ans.*

EXAMPLES. 83.

Find the value of

1. $\frac{2}{3}$ of ₨5. 7*a*. 6*p*.
2. $\frac{5}{6}$ of ₨2.
3. $\frac{3}{5}$ of ₨3 2*a*.
4. $\frac{4}{9}$ of ₨19. 3*a*. 6*p*.
5. $\frac{3}{8}$ of ₨3. 4*a*.
6. $\frac{5}{8}$ of 12*a*.
7. $\frac{4}{11}$ of £92 19*s*. 11*d*.
8. $\frac{5}{6}$ of £70 4*s*.
9. $\frac{3}{10}$ of £99.
10. $5\frac{3}{5}$ of ₨12. 9*a*. 8*p*.
11. ₨$\frac{7}{8}$ + ₨$\frac{11}{12}$.
12. ₨$2\frac{1}{2}$ − ₨$1\frac{1}{3}$.
13. $4\frac{3}{8}$ of £2. 11*s*. $7\frac{1}{4}d$.
14. $4\frac{4}{15}$ of £9.
15. $\frac{35}{112}$ of £1.
16. ₨13. 12*a*. 9*p*. $\times 3\frac{4}{5}$.
17. ₨13 13*a*. 6*p*. $\times 1\frac{5}{12}$.
18. £1. 7*s*. 6*d*. $\times\frac{27}{38}$.
19. £10. 10*s*. $10\frac{1}{2}d\times\frac{278}{189}$
20. ₨25. 12*a*. 9*p*. $\div\frac{66}{727}$.
21. £100. 3*s*. $4\frac{3}{4}d.\div 2\frac{9}{20}$ of $\frac{9}{5}$.
22. $3\frac{3}{4}$ of 1 cwt 1 qr. 1 lb.
23. $2\frac{2}{3}$ of 128 yd. 2 ft. 7 in.
24. $\frac{5}{12}$ of 1 hr. 1 min. 1 sec.
25. $\frac{4}{21}$ of 3 bus. 2 pk. 1 gali.
26. $3\frac{1}{2}$ of $3\frac{1}{3}$ of ₨12. 9*a*. 3*p*.
27. $\frac{1}{4}$ of $\frac{3}{7}$ of $1\frac{1}{3}$ of ₨7. 3*a*.
28. $2\frac{1}{3}$ of $6\frac{2}{5}$ of ₨7. 9*a* 3*p*. + $7\frac{1}{2}$ of ₨1. 3*a*. 4*p*.
29. $\frac{3}{4}$ of $4\frac{1}{3}$ of £2. 12*s*. 6*d*. − $\frac{5}{24}$ of £1. 6*s*. 6*d*.
30. £$7\frac{21}{40}+\frac{8}{9}$ of 15*s*. + 7*s*. $\div\frac{3}{4}+4\frac{2}{7}$ of £3. 3*s*.
31. ₨$13\frac{1}{6}-3\frac{7}{8}$ of 7*a*. − ₨2 4*a*. $\div\frac{99}{56}+7\frac{1}{9}$ of ₨3.
32. $\frac{436}{569}$ of ₨2 9*a*. + $\frac{117}{480}$ of ₨7. 8*a*. + $\frac{29}{338}$ of ₨9. 4*a*.
33. $\frac{3}{4}$ of $\frac{5}{8}$ of £1 + $\frac{2}{3}$ of $\frac{5}{9}$ of 2*s*. 6*d*. + $\frac{3}{4}$ of $10\frac{1}{2}d$.
34. $\frac{5}{6}$ of $\frac{7}{9}$ of ₨1 + $\frac{3}{4}$ of $\frac{5}{7}$ of 3*a* 9*p*. + $\frac{2}{3}$ of $7\frac{1}{2}p$.
35. $\frac{11}{12}$ of £1 + $\frac{3}{4}$ of 2 guineas − $\frac{5}{9}$ of 3*s*. 9*d*. + $\frac{7}{8}$ of 1*s*.
36. $\frac{3}{4}$ of a guinea + $\frac{5}{6}$ of a crown − $\frac{7}{8}$ of 3*s*. 6*d*.
37. $\frac{7}{8}$ of ₨7. 8*a*. 6*p*. − $\frac{4}{7}$ of 7*a*. 7*p*. + $\frac{2}{3\frac{1}{2}}$ of $\frac{4}{\frac{6}{7}-\frac{2}{9}}$ of ₨$\frac{5}{9}$.
38. $\frac{2\frac{3}{4}}{7-\frac{1}{8}}$ of ₨8. 9*a*. + $\frac{3\frac{7}{11}}{4\frac{2}{7}}$ of $\frac{10\frac{5}{7}}{7\frac{1}{2}}$ of ₨9. 0*a*. 7*p*.
39. $(3\frac{1}{4}\div3\frac{1}{3})$ of £3. 9*s*. $0\frac{3}{4}d$. + $(\frac{2}{3})^2$ of 27*s*. − $\frac{7\frac{3}{4}-3\frac{11}{12}}{18\frac{1}{2}\div\frac{6}{7}}$ of 5*s*.
40. Arrange $\frac{3}{4}$ of ₨7. $\frac{11}{12}$ of ₨6. 11*a*. and ₨$\frac{2}{3}$ in order of magnitude.
41. $\frac{3}{7}$ of $1\frac{1}{6}$ of a sum of money is £7. 7*s*. 7*d*. ; find the sum.

of the distances of two points of the image from the axis is not the same as the ratio of the corresponding points of the object; that is, when points far from the axis appear to be spread out too much or drawn in too much as compared with those near to the axis.

Angular Distortion is the distortion in which the angle between the distances of two points of the image from the axis is not the same as the corresponding angle for the object.

When an image is formed of centrical pencils, as the image that may be formed on a screen by a convex lens of which only a small portion near the axis is used, there is neither linear nor angular distortion.

Let us consider the distortion in the image of the object formed by a lens, seen by an eye on the axis of the lens. In this case the image seen will be formed by *excentrical* pencils, the rays coming to the eye through various points of the lens.

The linear distortion produced by a lens may be most easily explained by reference to its spherical aberration.

Suppose an eye placed at E on the axis of a convex lens to view the virtual image of an object A B C. Let us consider

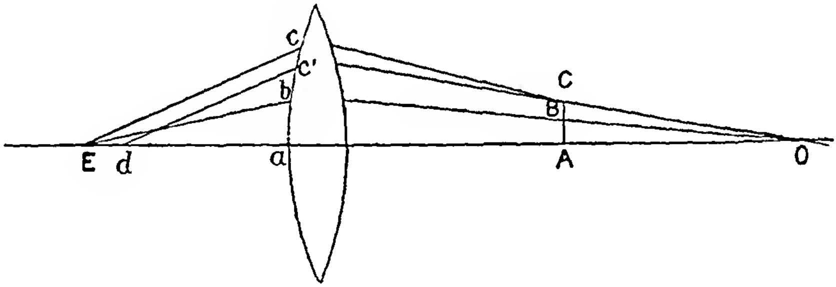


FIG. 93.

the paths of the rays that reach the eye from the various points of the object. Let O be the conjugate focus to E. O must be further from the lens than A is, because A is nearer than the principal focus, and O is further off. All rays which go to E would, on being produced backward, go through O but for spherical aberration. The ray B *b*, along O B produced, where B is a point close to the axis, goes to E. The ray C *c'*, along O C produced, meets the axis at a point, *d*, nearer to the lens than E. The ray from C which *does* reach E diverges more widely from the axis than C *c'* does. Let it be C *c* E. Thus, in the image which E sees of A B C, the images of B and C are on the productions of E *b* and E *c*. If they were along E *b* and E *c'*, A B and A C would appear magnified

proportionately. But, as it is, A C is more magnified than A B is. The parts of the object further away from the axis appear to be spread out too much from the axis, and there is linear distortion.

Similarly, the opposite sort of distortion produced in a real image by a convex lens may be explained as Fig. 94 shows.

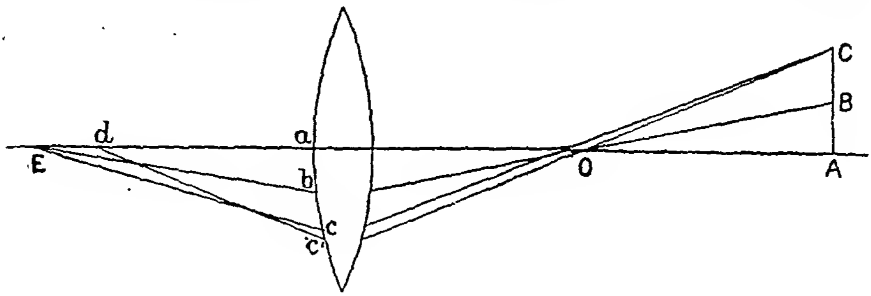


FIG. 94.

In this case O is nearer to the lens than A, because the conjugate focus to A is between E and the lens. The ray from C which reaches E meets the lens at a point, *c*, nearer to the axis than *c'* is. The images of B and C are seen along E *b*, E *c*. Thus the image of C is relatively nearer to the axis than that of B is.

Again, suppose an eye at E to view an object, A B C, through a concave lens. Let O be the focus conjugate to E.

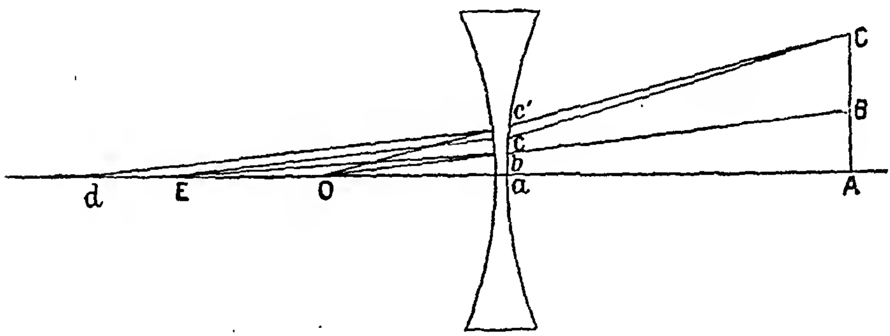


FIG. 95.

B being a point on the object very near to the axis, the ray B *b*, going to O, is deviated to E. The ray C *c'*, going to O, is deviated to *d*, a point further off than E. The ray from C, which *does* reach E, meets the lens at *c*, a point nearer to the axis than *c'* is. Thus in the image which E sees of A B C, the images of B and C are on E *b*, E *c* produced. Thus the image of C is relatively nearer to the axis than that of B is.

The linear distortion produced in the three cases may be illustrated as follows: Suppose, in each case, that the object

viewed is a square, the axis of the lens passing through its centre perpendicular to its plane. Then, in the first case—

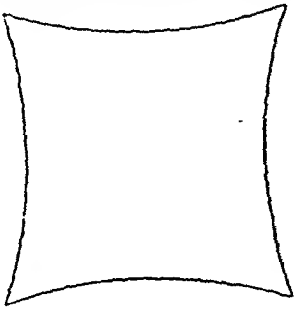


Fig. 96.

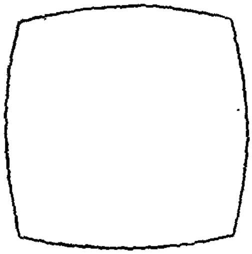


Fig. 97.

virtual image in convex lens—the image will have the appearance of Fig. 96; but in either of the other cases it will have the appearance of Fig. 97.

Curvature of Image formed by Excentrical Pencils.—The image, formed by excentrical pencils, of an object in a plane at right angles to the axis, that is, the image seen by an eye which views the object through the lens, will have curvature of the same sort in each case as that which exists in an image formed by centrical pencils, and already shown in Figs. 90, 91, 92. The explanation in this case is similar. Any lens with spherical surfaces acts, on account of spherical aberration, in its marginal portions as a lens of numerically smaller focal length than its proper focal length for direct centrical pencils. It was to this distortion that the well-known appearance due to linear distortion used to be attributed. Hence the latter distortion is generally called by its old name, want of **flatness of field.**

This distortion (of curvature in the image formed by excentrical pencils) is not of great importance; but it exists. The similar distortion of the image formed by centrical pencils is of much more importance; for it produces an effect on the image of a plane object formed on a screen; so that when the image of the central portion is formed distinctly, that of the marginal portion is not, and *vice versâ*. In the image which appears to the eye on looking through the lens it does not matter much if some parts are a little nearer than they ought to be, provided the parts keep their angular distances from the axis in the proper proportions; that is, provided there is no *linear* distortion.

Defects in the Images formed by Spherical Mirrors.—The defects of distortion in the images produced by spherical

mirrors will be similar to those in the images produced by lenses.

If the object is not all in one plane perpendicular to the axis, its dimensions perpendicular and parallel to the axis will not, in general, be magnified proportionately, the magnification at right angles to the axis being $\frac{v}{u}$, and parallel to the axis $\frac{v^2}{u^2}$.

If we have an object all in a plane at right angles to the axis of a spherical mirror, the area of the mirror being very limited, then of the pencils of light from the various points of the object, all, except that from the point on the axis, will, on reflexion, give rise to astigmatic pencils, producing more or less indistinct images of the points. Taking the circles of least confusion of these pencils as the image, the image of the plane object will be curved, as in the case of a lens.

Let us next consider linear distortion in a spherical mirror.

Suppose a concave spherical mirror produces a real image, *a b c*, of the object A B C, which image is seen by an eye at E

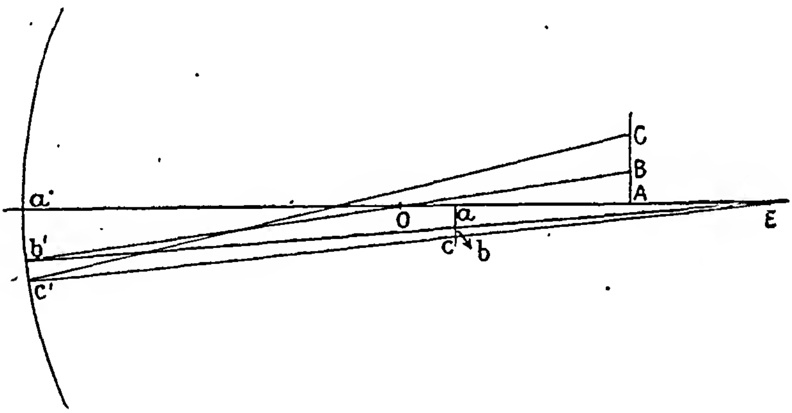


Fig. 98.

on the axis. Take O the conjugate focus to E. B, *b* being near the axis, the pencil from *b*, which reaches E, passed, before reflexion, through O. If there were no spherical aberration, the pencil from the point C, far from the axis, would also have come through O, and thus we should have $ac : ab = a'c' : a'b' = AC : AB$. But the pencil from *c* passed, before reflexion, through a point on the axis nearer to the mirror than O; thus we have $ac : ab < AC : AB$. And the image is contracted in the marginal parts.

Suppose we have a virtual image of A B C foımed in a concave mirror. The rays from *c* to E, before reflexion, came as

if from a point nearer the mirror than O; $ac : ab > AC : AB$ The image is expanded in the marginal parts.

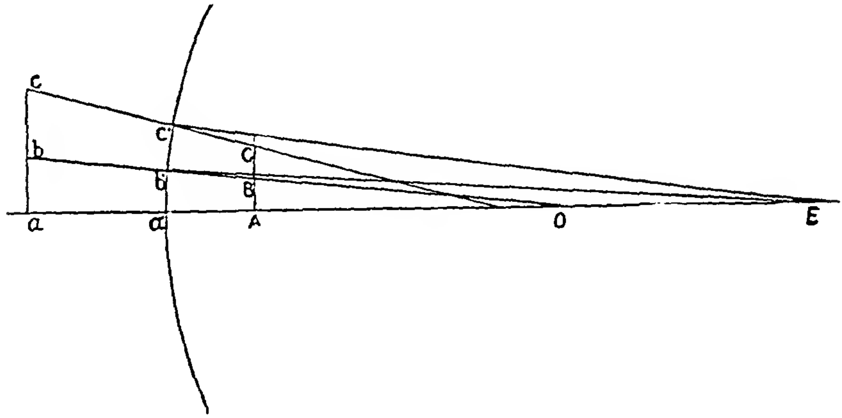


FIG. 99.

Suppose we have an image (virtual) of A B C formed in a convex mirror. The rays from *c* to E, before reflexion, were

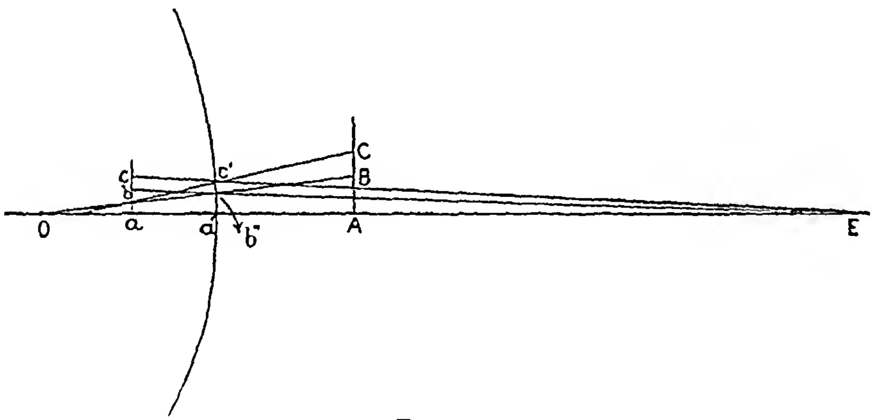


FIG. 100.

proceeding to a point nearer to the mirror than O. Thus $ac : ab < AC : AB$. The image is contracted in the marginal parts.

Notice that we have here the same results as for the corresponding cases with lenses, these being, respectively, real and virtual image with convex lens, and image (virtual) with concave lens.

CHAPTER VII.

THICK LENSES.

WE shall now consider the position of the image, formed by means of a thick lens, of a small object on the axis of the lens. Let the symbols have the following meanings:—

μ = refractive index of lens.
t = thickness of lens.
r = radius of front surface.
s = radius of back surface.
u = distance of object from front surface.
v = distance of image from back surface.
v' = distance of image formed by refraction at front surface from that surface.

Then we have, for the refractions at the two surfaces, the equations—

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{v} - \frac{\mu}{v' + t} = \frac{1 - \mu}{s} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From these we must eliminate v'.

Clearing of fractions, we may write the equations—

$$\mu ur - rv' - (\mu - 1)uv' = 0.$$

$$(v' + t)s - \mu vs + (\mu - 1)(v' + t)v = 0.$$

From these we get, by equating the two values they give for v',—

$$\frac{\mu ur}{r + (\mu - 1)u} = \frac{\mu vs - t\{s + (\mu - 1)v\}}{s + (\mu - 1)v}.$$

Multiplying up, and rearranging, we get—

$$(\mu - 1)\{\mu(r - s) + (\mu - 1)t\}uv + s\{\mu r + (\mu - 1)t\}u$$
$$- r\{\mu s - (\mu - 1)t\}v + trs = 0.$$

We can throw this into the form—

$$\frac{1}{v + \beta} - \frac{1}{u + \alpha} = \frac{1}{f}.$$

For this equation may be written—

$$uv - (f - \beta)u + (f + \alpha)v + f(\beta - \alpha) + \alpha\beta = 0.$$

Equating coefficients, we get—

$$\frac{1}{(\mu-1)\{\mu(r-s)+(\mu-1)t\}}=\frac{-f+\beta}{\mu rs+(\mu-1)st}$$
$$=\frac{-f-a}{\mu rs-(\mu-1)rt}=\frac{a\beta}{trs}.$$

These equations are satisfied by writing—

$$f=-\frac{\mu rs}{(\mu-1)\{\mu(r-s)+(\mu-1)t\}};$$

$$a=\frac{rt}{\mu(r-s)+(\mu-1)t};$$

$$\beta=\frac{st}{\mu(r-s)+(\mu-1)t}.$$

We see that the positions of object and image can be expressed by precisely the same formula as that used for a thin lens if u and v are measured, not from the surfaces, but from the points whose distances from the front and back surfaces are a and β. These points are called respectively the **first and second principal points** of the lens. The planes through them perpendicular to the axis are called the **principal planes.**

Magnification produced by Thick Lens.—We shall consider the magnifications produced at the surfaces. When a small object gives an image by refraction at a single spherical surface, corresponding points of object and image lie on the same radii. Thus the linear dimensions of object and image will be in the ratio of their distances from the centre. If, then, an object at distance u from the surface of radius r, refractive index μ, gives an image at distance v', the magnification is—

$$m=\frac{v'-r}{u-r}.$$

$$\text{But } \frac{\mu}{v'}-\frac{1}{\mu}=\frac{\mu-1}{r}.$$

$$\therefore \frac{\mu(v'-r)}{v'r}=\frac{u-r}{ur}.$$

$$\therefore m=\frac{v'}{\mu u}.$$

In the case of the thick lens this would be the magnification

produced at the first surface, and that produced at the second surface would be—

$$\frac{\mu v}{v'}.$$

Thus the magnification produced by the lens is—

$$\frac{v}{u}.$$

Let H, H′ be the principal points of a thick lens; P and Q a pair of conjugate foci. Let a pair of corresponding rays through P and Q meet the principal planes in the points R

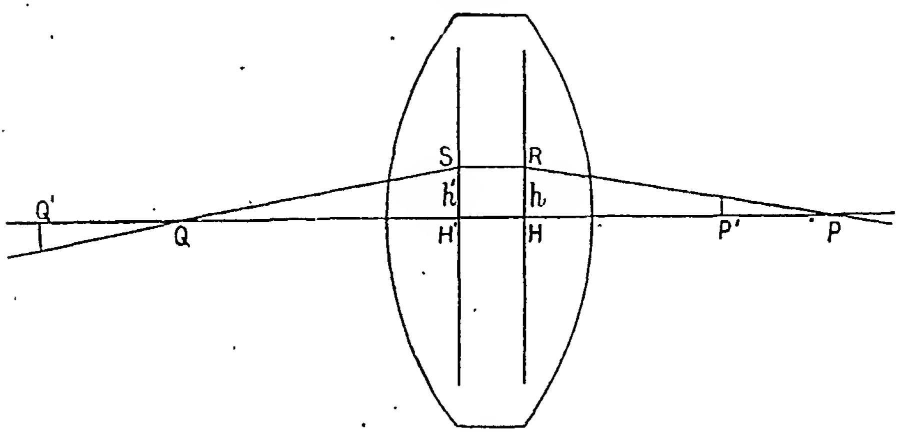


FIG. 101.

and S, at distances h and h' from the axis. We can find the relation between h and h' by the following device: A small object, P′, will give rise to an image at a known point, Q′. Now, we know the relation of size of image to size of object: so that if we take PR as a ray proceeding from a point of the object, we know how the corresponding ray, S Q, must be drawn, for we know through what point of the image to draw it.

Let the distances of P, P′ from H be u, u'; and those of Q, Q′ from H′ be v, v'. Let β, β' be the lengths of object and image. Then—

$$\frac{h}{u} = \frac{\beta}{u - u'},$$

$$\frac{h'}{v} = \frac{\beta'}{v - v'}.$$

And—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} = \frac{1}{v'} - \frac{1}{u'};$$

$$\therefore \frac{u - u'}{uu'} = \frac{v - v'}{vv'};$$

$$\frac{h}{h'} = \frac{\beta}{\beta'} \cdot \frac{u}{v} \cdot \frac{v - v'}{u - u'}$$

$$= \frac{\beta}{\beta'} \cdot \frac{v'}{u'} = 1.$$

$$h = h'.$$

That is, corresponding rays, before entry and at emergence, meet the principal planes in points equidistant from the axis.

This gives a useful method of drawing the paths of corresponding rays. Suppose a ray through the point P on the axis to cut the axis again at Q after passing through the lens. Draw PR to meet the first principal plane in R. Draw R S parallel to the axis to meet the second principal plane in S. Then S Q is the path of the emergent ray.

Suppose an object in the first principal plane. This may be a real object if the plane is outside the lens; or may be a virtual object formed by converging rays in any case. The image will be in the second principal plane. For if in the equation—

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

we make $u = 0$, then $v = 0$. By what we have just seen, the rays through any point of the object diverge on emerging as if they came from a point at the same distance from the axis. Thus the image is of the same size as the object. The principal planes are called, in consequence, **planes of unit magnification.**

We may also prove these properties of the principal planes as follows: u and v being distances of image and object, measured from the principal points, the magnification is $\frac{v}{u}$.

$$\text{But since } \frac{1}{v} - \frac{1}{u} = \frac{1}{f},$$

$$\frac{v}{u} = 1 - \frac{v}{f}.$$

Therefore when u and v vanish, the magnification becomes unity.

It follows that a ray going before entry to any point on the first principal plane, proceeds on emergence from a point at an equal distance from the axis on the second principal plane. That is, the line R S, in the above figure, is parallel to the axis.

It should be noticed that the principal points are a pair of conjugate foci, since when $u = 0$, $v = 0$.

A ray proceeding to the first principal point emerges from the second parallel to its original direction. For if Pp, Qq are object and image, $Pp : Qq = HP : H'Q$. Thus pH and H'q are parallel.

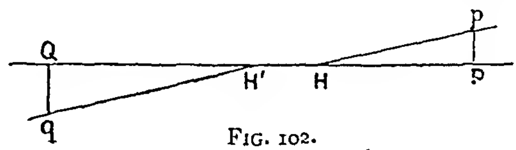


FIG. 102.

From what we have seen, it follows that graphic constructions are to be made in the same way as for thin lenses, only with the addition of the space between the principal planes. If this space were removed, the points H, H′ would coincide; and so would the points R and S in the last figure.

If the media on the two sides of a lens are any whatever, not necessarily the same, a pair of conjugate foci can be found such that if a ray proceeds towards one before entering the lens, it emerges from the other in a direction parallel to its original direction. These points are called **nodal points**. When the lens is situated in air, the nodal points coincide with the principal points.

The formula for the focal length has been given above; the principal focus is at this distance from the second principal point; and just as for a thin lens, there is another principal focus, for rays coming in the opposite direction. This is at a distance which is numerically the same from the first principal focus.

Except so far as it involves t, the thickness, the focal length will be just the same as for a thin lens of the same substance and the same curvatures. When t is small, the focal length will, as a rule, be positive for a lens thinnest in the middle, and negative for a lens thickest in the middle. But it may happen that t may decide the sign of f. For suppose $r - s$ negative, and numerically less than $\dfrac{t(\mu - 1)}{\mu}$. The focal length of the lens is then of opposite sign to that of a thin lens of the same substance and curvatures.

Suppose a lens has equal and opposite curvatures, so that $r = -s$; has small thickness; and is made of crown glass,

142. Division of Decimals.

I. *When the Divisor is an Integer:*

Example 1. Divide 808·9 by 25.

Process:

```
25 ) 808·9 ( 32·356   Ans.
     75
      58
      50
       89
       75
       140
       125
        150
        150
```

Here we divide as in the case of whole numbers, taking care to place the decimal point in the quotient as soon as the division of the integral part is finished.

If there is a remainder (as in the above case) after division, we affix a zero to the remainder, and divide. We treat all successive remainders in the same manner, and continue the division until the required number of decimal places in the quotient is obtained, or until there is no remainder.

Note. The method of short division may be employed with advantage when the divisor does not exceed 20, or when the divisor can be expressed as the product of factors each less than 20.

Example 2. Obtain the quotient to five places of decimals in the division of ·025 by 7.

Process:

```
7 ) ·025
    ·00357...   Ans.
```

II. *When the Divisor is a decimal:*

Remove the decimal point in both the Divisor and Dividend as many places to the right as will make the *divisor* a whole number; and then divide as in the preceding case.

Note. Observe that removing the decimal point in the divisor and dividend an equal number of places to the right is equivalent to multiplying the divisor and dividend by the same number; and that if the divisor and dividend be both multiplied by the same number the quotient is not altered.

Example 3. Divide 12·96 by 10·8.

Here we divide 129·6 by 108 :

108) 129·6 (1·2 *Ans.*
108
216
216

Example 4. Divide 34·6 by ·08.

Here we divide 3460· by 8 :

8) 3460·
432·5 *Ans.*

143. A vulgar fraction may be expressed as a decimal by dividing the numerator by the denominator.

Example. Express $\frac{5}{8}$ as a decimal.

Process : 8) 5·
·625 *Ans.*

Note. The following results are useful :

$\frac{1}{2}=$·5 ; $\frac{1}{4}=$·25 ; $\frac{3}{4}=$·75 ; $\frac{1}{8}=$·125.

EXAMPLES. 91.

Divide

1. 29·21 by 23. **2.** 34·3 by 25. **3.** 129·6 by 108.

4. ·03096 by 72. **5.** 457·7 by 230. **6.** ·06227 by 1300.

7. ·04009 by 1520. **8.** 3708 by 360. **9.** ·00281 by 1405.

10. 8357 by 488. **11.** ·001007 by 47500. **12.** 431·376 by 8170.

Divide, finding the quotient as far as the fifth decimal place:

13. 42·5 by 23. **14.** ·0269 by 281. **15.** 197 by 79.

16. ·041326 by 101. **17.** ·0079 by 372. **18.** 312 by 84.

19. 356·5 by 273. **20.** 6·5 by 342. **21.** ·0042 by 121.

Find the quotient, by Short Division, to not more than 6 places of decimals, in the division of

22. 4·125 by 2. **23.** 3·73 by 8. **24.** ·034 by 7.

25. 21·24 by 90. **26.** 134 by 11. **27.** 36·7 by 16.

28. ·04321 by 80. **29.** 8·567 by 13. **30.** ·01 by 6.

Divide

31. ·3125 by ·01. **32.** 8·454 by ·024. **33.** ·5568 by 2·32.

34. 6·33 by ·0025. 35. 17·28 by ·0144. 36. 4 by ·00625.
37. ·00281 by 1·405. 38. 1·77089 by 4·735.
39. ·00005 by ·0000025. 40. 816 by ·0004.
41. 84·375 by ·00375. 42. 2874·465 by ·0495.
43. ·830676 by ·000231. 44. 33·363 by ·00275.
45. 7 by ·0004. 46. ·0007 by ·0005.
47. 5·625 by ·0000075. 48. ·0003738028 by ·0476.

Find the quotient to five places of decimals :

49. 3·461 ÷ ·027. 50. ·3125 ÷ ·06.
51. ·2 ÷ ·006. 52. ·000753 ÷ ·009.
53. ·000001 ÷ ·0000431. 54. ·5 ÷ 76·91342.
55. 4000 ÷ ·000121. 56. ·666666 ÷ ·008.
57. ·007 ÷ ·00073. 58. 4·00654 ÷ 329·265.

Employ Short Division in finding the quotient to not more than 6 places of decimals :

59. 28 ÷ ·08. 60. 3·76 ÷ ·005. 61. ·0076 ÷ ·003.
62. ·0101 ÷ ·0016. 63. ·000012 ÷ ·13. 64. 229 ÷ ·007.
65. 39·4 ÷ ·007. 66. 4·767 ÷ ·004. 67. 13·75 ÷ ·012.
68. ·02 ÷ 1·1. 69. ·03 ÷ 1·4. 70. 3·4 ÷ ·009.

Simplify

71. $\dfrac{\cdot0075 \times 2\cdot1}{\cdot0175}$. 72. $\dfrac{1\cdot18}{\cdot152} \times \dfrac{3\cdot04}{2\cdot95}$. 73. $\dfrac{\cdot081 \times 5\cdot7}{1\cdot71}$.

Convert into decimals :

74. $\frac{1}{2}$. 75. $\frac{1}{4}$. 76. $\frac{3}{4}$. 77. $\frac{1}{8}$. 78. $\frac{3}{8}$.
79. $1\frac{7}{16}$. 80. $3\frac{3}{32}$. 81. $9\frac{21}{24}$. 82. $3\frac{7}{25}$. 83. $\frac{201}{16}$.

Express as decimals as far as the fifth decimal place :

84. $\frac{1}{3}$. 85. $\frac{1}{6}$. 86. $\frac{2}{7}$. 87. $\frac{3}{11}$. 88. $\frac{9}{13}$.
89. $1\frac{4}{9}$. 90. $7\frac{2}{11}$. 91. $8\frac{7}{21}$. 92. $10\frac{10}{29}$. 93. $\frac{201}{12}$.

Arrange in order of magnitude, by reducing to decimals as far as the fourth decimal place :

94. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}$. 95. $\frac{3}{11}, \frac{5}{12}, \frac{7}{14}$. 96. $\frac{11}{20}, \frac{16}{30}, \frac{21}{40}$.
97. $\frac{5}{16}, \frac{7}{32}, \frac{3}{8}$. 98. $\frac{7}{20}, \frac{11}{25}, \frac{13}{30}$. 99. $\frac{3}{5}, \frac{5}{7}, \frac{7}{9}$.

Reduce to decimals :

100. $\frac{4}{5}$ of ·027. 101. ·025 of $4\frac{1}{2}$.
102. $\frac{1}{2}$ of $\frac{3}{4} \times 8\cdot36$. 103. $\frac{1}{4}$ of $\frac{1}{10} \div \cdot05$ of $2\frac{1}{2}$.

144. H. C. F. and L. C. M. of Decimals.

To find the H. C. F. or the L. C. M. of Decimals, affix ciphers (where necessary) so that all the given numbers may have the same number of decimal places; then find the H. C. F. or the L. C. M. of them as if they were integers, and mark off in the result as many decimal places as there are in each of the numbers.

Example. Find the H. C. F. and L. C. M. of 3, 1·2 and ·06.

The given numbers are equivalent to 3·00, 1·20 and ·06.

The H. C. F. of 300, 120 and 6 = 6; their L. C. M. = 600.

∴ The H. C. F. required = ·06;

and the L. C. M. required = 6·00 = 6.

EXAMPLES. 92.

Find the H. C. F. and L. C. M. of

1.	3·75, 7·25.	**2.**	72·12, ·03.	**3.**	·02, ·4, ·008.
4.	1·2, ·24, 6.	**5.**	1·6, ·04, ·005.	**6.**	2·4, ·36, 7·2.
7.	·08, ·002, ·0001.	**8.**	3·9, 6·6, 8·22.	**9.**	·6, ·09, 1·8.
10.	·18, 2·4, 60.	**11.**	20, 2·8, ·25.	**12.**	1·5, ·25, ·075.

XXVI. RECURRING DECIMALS.

145. In the process of reduction of vulgar fractions to decimals, it will be found, in some cases, that the division does not terminate; so that the quotient can be continued without limit.

Example. Reduce $\frac{19}{55}$ to a decimal.

$$55\;)\;\underline{19\cdot\qquad\qquad}$$
$$\cdot3454545\ldots$$

146. We can tell beforehand whether, in any particular case, the division will terminate or not.

Let the fraction be in its lowest terms; then if the prime factors of the denominator are each of them either 2 or 5, the division will terminate; and not otherwise.

Thus

(i) $\frac{7}{20} = \left(\frac{7}{2\times2\times5}\right)$ will produce a **terminating** decimal.

(ii) $\frac{7}{12} = \left(\frac{7}{2\times2\times3}\right)$ will produce a **non-terminating** decimal.

EXAMPLES. 93.

State, in each case, whether the equivalent decimal is terminating or non-terminating :

1. $\frac{1}{3}$. 2. $\frac{3}{4}$. 3. $\frac{7}{8}$. 4. $\frac{21}{24}$. 5. $\frac{76}{29}$.
6. $2\frac{49}{63}$. 7. $\frac{33}{121}$. 8. $\frac{12}{175}$. 9. $\frac{34}{510}$. 10. $\frac{101}{135}$.
11. $3\frac{18}{72}$. 12. $\frac{9}{45}$. 13. $7\frac{1}{2}$. 14. $\frac{72}{96}$. 15. $11\frac{10}{12}$.

16. Write down those numbers between 1 and 20, which being denominators of fractions in their lowest terms, will produce non-terminating decimals.

147. In non-terminating decimals, certain digits must recur over and over again.

Consider the fraction $\frac{5}{6}$. In the process of division the only remainders possible are 1, 2, 3, 4, 5 ; consequently, after five steps at most, we must come to a remainder which has occurred before, and therefore from that point we must have a recurrence of the remainders, and therefore of the digits in the quotient.

Example 1. $\frac{2}{3} = \cdot 6666666...$

Example 2. $\frac{19}{55} = \cdot 3454545...$

Note. It may be noticed here that division by 3 or 9 gives a period (See Art. 148) of *one* digit : division by 11, a period of *two* digits ; division by 7 or 13, a period of *six* digits.

148. Decimals in which certain digits recur are called **recurring decimals.**

Note. A recurring decimal is also called a **periodic, repeating** or **circulating** decimal.

The whole body of digits which recur is called the **period.** Thus, in ·6666...the period is 6 ; in ·3454545...the period is 45.

149. In writing a recurring decimal we usually stop at the end of the first period and place dots over its first and last digits.

Thus ·666666......... is written $\cdot\dot{6}$;

·373737......... .... $\cdot\dot{3}\dot{7}$;

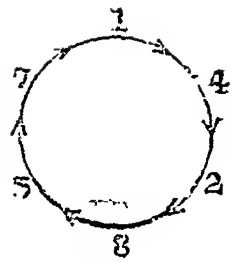
·3454545......... .... $\cdot 3\dot{4}\dot{5}$;

·34576576...... .... $\cdot 34\dot{5}7\dot{6}$.

A **pure** recurring decimal is one in which the period commences immediately after the decimal point ; as, $\cdot\dot{6}$, $\cdot\dot{3}\dot{7}$.

A **mixed** recurring decimal is one in which one or more figures precede the period ; as, $\cdot 3\dot{4}\dot{5}$, $\cdot 34\dot{5}7\dot{6}$

Note. It may be noticed that decimals equivalent to fractions with denominator 7 are all *pure* recurring decimals, all of which contain the same digits 142857. If these digits be arranged in a circle, as in the annexed diagram, we may obtain the decimals equivalent respectively to $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$, by beginning in turn with 1, 2, 4, 5, 7, 8, and reading off the remaining digits in order in the direction of the arrow-heads.



Thus $\frac{1}{7} = \cdot\dot{1}4285\dot{7}$; $\frac{2}{7} = \cdot\dot{2}8571\dot{4}$; $\frac{3}{7} = \cdot\dot{4}2857\dot{1}$; and so on.

EXAMPLES. 94.

Express each of the following as a recurring decimal:

1. $\frac{1}{3}$.	2. $\frac{2}{9}$.	3. $\frac{5}{7}$.	4. $\frac{7}{6}$.	5. $\frac{13}{11}$.
6. $\frac{20}{13}$.	7. $\frac{7}{15}$.	8. $1\frac{1}{111}$.	9. $\frac{5}{18}$.	10. $3\frac{3}{13}$.
11. $\frac{250}{21}$.	12. $\frac{1}{22}$.	13. $\frac{20000}{5291}$.	14. $\frac{5}{24}$.	15. $\frac{101}{26}$.
16. $\frac{202}{27}$.	17. $5\frac{2}{7}$.	18. $10\frac{1}{13}$.	19. $7\frac{2}{15}$.	20. $9\frac{9}{14}$.
21. $\frac{506}{505}$.	22. $\frac{725}{52}$.	23. $4\frac{45}{56}$.	24. $\frac{3542}{1025}$.	25. $5\frac{11}{90}$.
26. $2 \div 3$.	27. $46 \div 7$.	28. $39 \div 22$.	29. $8 \div 63$.	30. $44 \div 9$.
31. $\frac{1}{6}$.	32. $\frac{1}{66}$.	33. $\frac{1}{666}$.	34. $\frac{1}{6666}$.	35. $\frac{1}{66666}$.
36. $8\frac{8}{75}$.	37. $3\frac{24}{175}$.	38. $\frac{1}{17}$.	39. $\frac{40}{19}$.	40. $\frac{2}{23}$.

41. $1\cdot2 \div \cdot11$.	42. $1 \div 10\cdot01$.	43. $\cdot3 \div \cdot13$.	44. $\frac{\cdot2}{\cdot07}$.
45. $\frac{\cdot03}{\cdot0011}$.	46. $2 + \frac{\cdot3}{1\cdot1}$.	47. $7 + \frac{2}{2\cdot3}$.	48. $1 + \frac{1\cdot1}{\cdot07}$.
49. $3 + \frac{4}{1\cdot3}$.	50. $\frac{4\frac{1}{2}}{\cdot007}$.	51. $\frac{3\cdot7}{4\frac{1}{2}}$.	52. $\frac{\cdot004}{5\frac{1}{2}}$.

150. In a given recurring decimal, the period may be supposed to begin at any point after the first repeating figure.

Thus $\cdot327272727\ldots = \cdot3\dot{2}\dot{7} = \cdot32\dot{7}\dot{2} = \cdot327\dot{2}\dot{7} =$ etc.

Again, the number of figures in the period of a recurring decimal may be *doubled*, *trebled*,...without altering the value of the decimal.

Thus $\cdot3\dot{2}\dot{7} = \cdot3\dot{2}72\dot{7} = \cdot3\dot{2}7272\dot{7} =$ etc.

151. Recurring decimals are said to be **similar** when they have the same number of non-recurring figures, and also the same

number of recurring figures. Thus $\cdot\dot{3}$ and $\cdot\dot{6}$ are similar recurring decimals ; $\cdot 3\dot{2}\dot{7}$ and $2\cdot 4\dot{5}\dot{6}$ are similar.

152. Two or more given recurring decimals can always be made similar.

Take the recurring decimals $2\cdot\dot{3}$, $\cdot 2\dot{4}\dot{5}$ and $\cdot 25\dot{7}6\dot{8}$.

Now the highest number of non-recurring decimal places in any of these numbers is 2 ; and the numbers of figures in the periods respectively are 1, 2, 3, the L. C. M. of which is 6. Therefore the given recurring decimals may be made similar by extending each of them to eight places of decimals, the first two places being non-recurring and the last six places being recurring.

Thus
$$2\cdot\dot{3} = 2\cdot 33\dot{3}3333\dot{3};$$
$$\cdot 2\dot{4}\dot{5} = \cdot 24\dot{5}4545\dot{4};$$
$$\cdot 25\dot{7}6\dot{8} = \cdot 25\dot{7}6876\dot{8}.$$

EXAMPLES. 95.

In each of the following recurring decimals begin the period at the fourth decimal place :

1. $\cdot 2\dot{3}4\dot{5}$.
2. $\cdot 34\dot{7}\dot{6}$.
3. $\cdot\dot{6}\dot{7}$.
4. $\cdot 23\dot{4}\dot{5}$.
5. $\cdot 00\dot{1}2\dot{3}$.
6. $\cdot 1\dot{2}34\dot{5}$.
7. $\cdot\dot{1}23\dot{4}$.
8. $\cdot 1\dot{2}345\dot{6}$.
9. Extend $\cdot 3\dot{4}$, $\cdot\dot{2}\dot{4}$ and $\cdot 2\dot{6}7\dot{8}$ so that they may have the same number of figures in the period.
10. Extend $\cdot 1\dot{0}\dot{2}$, $\cdot 1\dot{2}3\dot{4}$ and $\cdot\dot{3}76\dot{5}$ so that they may have the same number of recurring figures.

Make the following sets of recurring decimals similar :

11. $\cdot 2\dot{3}$, $\cdot\dot{7}\dot{8}$.
12. $\cdot 3\dot{4}\dot{5}$, $\cdot\dot{7}\dot{6}$, $\cdot 7\dot{2}$.
13. $\cdot 30\dot{7}$, $\cdot\dot{7}\dot{6}$.
14. $\cdot 0\dot{7}\dot{6}$, $\cdot\dot{7}$, $\cdot 00\dot{0}12\dot{3}$.
15. $\cdot 23\dot{8}$, $\cdot\dot{1}23\dot{4}$, $\cdot 0\dot{2}\dot{3}$.
16. $\cdot\dot{3}$, $\cdot\dot{7}\dot{6}$, $\cdot\dot{7}23\dot{0}$.
17. $\cdot\dot{7}$, $\cdot 1\dot{2}\dot{4}$, $\cdot 24\dot{7}2\dot{3}$.
18. $3\cdot\dot{4}$, $\cdot 2\dot{6}\dot{8}$, $\cdot\dot{1}2\dot{3}$.
19. $3\cdot 40\dot{2}$, $\cdot 78\dot{2}\dot{3}$, $\cdot 3\dot{1}$.
20. $\cdot 4\dot{2}\dot{3}$, $\cdot\dot{7}\dot{2}$, $\cdot 1\dot{2}0\dot{3}$.

153. To express a recurring decimal as a vulgar fraction.

Example 1. $\cdot\dot{5} = \cdot 55555\ldots$

Now, 10 times $\cdot\dot{5} = 5\cdot 5555\ldots$

and $\cdot\dot{5} = \cdot 5555\ldots$

Subtracting, 9 times $\cdot\dot{5} = 5$;

$\therefore \cdot\dot{5} = \frac{5}{9}$.

Example 2. $\cdot 23\dot{4}\dot{5}=\cdot 23454545\ldots$

Now, 10000 times $\cdot 23\dot{4}\dot{5}=2345\cdot 4545\ldots$

and 100 times $\cdot 23\dot{4}\dot{5}=23\cdot 4545\ldots$

Subtracting, 9900 times $\cdot 23\dot{4}\dot{5}=2345-23$;

$\therefore\ \cdot 23\dot{4}\dot{5}=\frac{2345-23}{9900}$.

Example 3. $3\cdot 6\dot{2}=3\cdot 622222\ldots$

Now, 100 times $3\cdot 6\dot{2}=362\cdot 2222\ldots$

and 10 times $3\cdot 6\dot{2}=36\cdot 2222\ldots$

Subtracting, 90 times $3\cdot 6\dot{2}=362-36$;

$\therefore\ 3\cdot 6\dot{2}=\frac{362-36}{90}$.

154. Hence we deduce the following rule for reducing a recurring decimal to a vulgar fraction :

For the *numerator* take the integral number formed by all the figures up to the end of the first period, subtracting the integral number formed by the figures (if any) that precede the first period ; for the *denominator* take the number formed by as many nines as there are figures in the period, followed by as many ciphers as there are figures between the decimal point and the first period.

Example 1. Find the vulgar fraction equivalent to $\cdot\dot{3}$.

Process : $\cdot\dot{3}=\frac{3}{9}=\frac{1}{3}$. *Ans.*

Example 2. Reduce $\cdot 4\dot{5}$ to vulgar fraction.

Process : $\cdot 4\dot{5}=\frac{45-4}{90}=\frac{41}{90}$. *Ans.*

Example 3. Express $\cdot 04\dot{7}\dot{6}$ as a vulgar fraction.

Process : $\cdot 0\cdot 4\dot{7}\dot{6}=\frac{476-4}{9900}=\frac{472}{9900}=\frac{118}{2475}$. *Ans.*

Example 4. Express $\cdot 00\dot{2}7\dot{1}$ as a vulgar fraction.

Process : $\cdot 00\dot{2}7\dot{1}=\frac{271}{99900}$. *Ans.*

Example 5. Express $2\cdot 3\dot{7}$ as an improper fraction.

Process : $2\cdot 3\dot{7}=\frac{237-23}{90}=\frac{214}{90}=\frac{107}{45}$. *Ans.*

Example 6. Express $2\cdot 3\dot{7}$ as a mixed number.

Process : $2\cdot 3\dot{7}=2+\cdot 3\dot{7}=2+\frac{37-3}{90}=2+\frac{17}{45}=2\frac{17}{45}$. *Ans.*

Note. It follows from the rule that $\cdot\dot{9}=\frac{9}{9}=1$; similarly $\cdot 0\dot{9}=\cdot 1$ and $\cdot 00\dot{9}=\cdot 01$; and therefore $2\cdot\dot{9}=3$, $2\cdot 3\dot{9}=2\cdot 4$, $2\cdot 345\dot{9}=2\cdot 346$; etc. Also $\cdot\dot{9}\dot{9}=1$, $\cdot\dot{9}9\dot{9}=1$, $\cdot 2\dot{9}\dot{9}=\cdot 3$; etc.

Therefore when the recurring part contains the figure 9 *only*, the recurring part should be omitted and the preceding figure increased by unity.

EXAMPLES. 96.

Express as vulgar fractions in their lowest terms :

1. $\cdot\dot{6}$. 2. $\cdot\dot{1}\dot{8}$. 3. $\cdot\dot{1}4285\dot{7}$. 4. $\cdot\dot{7}6923\dot{0}$.
5. $\cdot2\dot{7}$. 6. $\cdot27\dot{2}$. 7. $\cdot3\dot{7}\dot{8}$. 8. $\cdot03\dot{2}$.
9. $\cdot00\dot{7}8\dot{5}$. 10. $\cdot008\dot{2}\dot{3}$. 11. $\cdot0010\dot{5}\dot{4}$. 12. $\cdot0\dot{8}\dot{1}$.
13. $3\cdot\dot{0}1\dot{3}$. 14. $3\cdot4\dot{3}\dot{2}$. 15. $7\cdot0\dot{2}\dot{8}$. 16. $31\cdot00\dot{7}$.
17. $\cdot5\dot{9}2\dot{5}$. 18. $\cdot0\dot{5}$. 19. $2\cdot\dot{6}1904\dot{7}$. 20. $10\cdot25\dot{6}\dot{7}$.
21. $\cdot0012\dot{3}$. 22. $\cdot011\dot{3}\dot{6}$. 23. $\cdot00\dot{2}\dot{9}$. 24. $\cdot38\dot{1}4\dot{8}$.
25. $\cdot00\dot{6}7\dot{5}$. 26. $\cdot0\dot{2}\dot{4}$. 27. $\cdot\dot{0}37\dot{8}$. 28. $\cdot2\dot{2}7\dot{3}$.
29. $\cdot00\dot{0}2\dot{5}$. 30. $\cdot100\dot{0}\dot{1}$. 31. $3\cdot0\dot{2}0\dot{7}$. 32. $\cdot02\dot{1}7\dot{7}$.

Reduce to improper fractions in their lowest terms :

33. $3\cdot\dot{6}$. 34. $7\cdot\dot{1}\dot{8}$. 35. $1\cdot3\dot{1}$. 36. $2\cdot7\dot{6}$.
37. $1\cdot07\dot{2}$. 38. $3\cdot03\dot{6}$. 39. $10\cdot2\dot{7}\dot{5}$. 40. $4\cdot0\dot{0}8\dot{6}$.
41. $7\cdot12\dot{3}\dot{0}$. 42. $7\cdot6\dot{5}3\dot{1}$. 43. $20\cdot45\dot{0}\dot{0}$. 44. $14\cdot0\dot{1}3\dot{1}$.
45. $10\cdot02\dot{2}\dot{7}$. 46. $13\cdot94\dot{3}076\dot{9}$. 47. $11\cdot00\dot{1}20\dot{0}$. 48. $100\cdot00\dot{1}0\dot{0}$.

49. Prove that $\dfrac{1}{9}=\dfrac{\cdot\dot{1}}{1}=\dfrac{\cdot\dot{2}}{2}=\dfrac{\cdot\dot{3}}{3}=\dfrac{\cdot\dot{4}}{4}=\dfrac{\cdot\dot{5}}{5}=\dfrac{\cdot\dot{6}}{6}=\dfrac{\cdot\dot{7}}{7}=\dfrac{\cdot\dot{8}}{8}$.

50. Prove that $\dfrac{1}{11}=\dfrac{\cdot\dot{0}\dot{9}}{1}=\dfrac{\cdot\dot{1}\dot{8}}{2}=\dfrac{\cdot\dot{2}\dot{7}}{3}=\dfrac{\cdot\dot{3}\dot{6}}{4}=\dfrac{\cdot\dot{4}\dot{5}}{5}=\dfrac{\cdot\dot{5}\dot{4}}{6}$.

51. Prove that $\dfrac{1}{13}=\dfrac{\cdot\dot{0}7692\dot{3}}{1}=\dfrac{\cdot\dot{1}5384\dot{6}}{2}=\dfrac{\cdot\dot{2}3076\dot{9}}{3}=\dfrac{\cdot\dot{3}0769\dot{2}}{4}$.

52. Prove that $\dfrac{\cdot\dot{1}0\dot{1}}{1}=\dfrac{\cdot\dot{2}0\dot{2}}{2}=\dfrac{\cdot\dot{3}0\dot{3}}{3}=\dfrac{\cdot\dot{4}0\dot{4}}{4}=\dfrac{\cdot\dot{5}0\dot{5}}{5}$.

Express as non-recurring decimals :

53. $\cdot0\dot{9}$. 54. $\cdot357\dot{9}$. 55. $1\cdot6\dot{9}$. 56. $\cdot000\dot{9}$.
57. $\cdot2\dot{9}\dot{9}$. 58. $3\cdot\dot{9}\dot{9}$. 59. $3\cdot9\dot{9}\dot{9}$. 60. $9\cdot\dot{9}9\dot{9}$.

155. Addition and Subtraction of Recurring Decimals.

Rule for Addition : Make the decimals *similar* : add in the usual way and *increase* the last figure in the result by the figure (if any) carried from the first (to the left) column of the period ; then the sum will be a recurring decimal similar to the summands.

Subtraction is effected in exactly the same way, the only difference being that the last figure in the result in this case is diminished (and not increased) by the figure carried.

Example 1. Add together $2\cdot3\dot{7}\dot{5}$, $\cdot8\dot{1}7\dot{3}$ and $4\cdot31$.

Process :

$$\begin{array}{rl}
2\cdot3\dot{7}\dot{5} & = 2\cdot37\ \dot{5}7575\dot{7} \\
\cdot8\dot{1}7\dot{3} & = \ \ \cdot81\ \dot{7}3173\dot{1} \\
4\cdot31 & = 4\cdot31 \\
\hline
 & \ \ \ 7\cdot50\ 307488 \\
 & \qquad\qquad\quad 1 \\
\hline
 & \ \ \ 7\cdot50\ \dot{3}0748\dot{9} \quad \textit{Ans.}
\end{array}$$

Example 2. Add together $7\cdot63\dot{4}$ and $\cdot8\dot{5}\dot{2}$.

Process :

$$\begin{array}{rl}
7\cdot63\dot{4} & = 7\cdot63\ \dot{4}\dot{4} \\
\cdot8\dot{5}\dot{2} & = \ \ \cdot85\ \dot{2}\dot{5} \\
\hline
 & \ \ 8\cdot48\ \dot{6}\dot{9} \quad \textit{Ans.}
\end{array}$$

Example 3. Add together $\cdot76\dot{8}$, $\cdot0\dot{7}$ and $1\cdot0\dot{3}$.

Process :

$$\begin{array}{rl}
\cdot76\dot{8} & = \ \ \cdot76\ \dot{8} \\
\cdot0\dot{7} & = \ \ \cdot07\ \dot{7} \\
1\cdot0\dot{3} & = 1\cdot03\ \dot{3} \\
\hline
 & \ 1\cdot87\ 8 \\
 & \qquad\ \ 1 \\
\hline
 & \ 1\cdot87\ \dot{9} = 1\cdot88 \quad \textit{Ans.}
\end{array}$$

Example 4. Subtract $\cdot78\dot{3}7\dot{2}$ from $4\cdot0\dot{7}\dot{1}$.

Process :

$$\begin{array}{rl}
4\cdot0\dot{7}\dot{1} & = 4\cdot07\ \dot{1}7171\dot{7} \\
\cdot78\dot{3}7\dot{2} & = \ \ \cdot78\ \dot{3}7237\dot{2} \\
\hline
 & \ 3\cdot28\ 799345 \\
 & \qquad\qquad\ \ 1 \\
\hline
 & \ 3\cdot28\ \dot{7}9934\dot{4} \quad \textit{Ans.}
\end{array}$$

Example 5. Subtract $\cdot8\dot{6}\dot{2}$ from $6\cdot74\dot{5}$.

Process :

$$\begin{array}{rl}
6\cdot74\dot{5} & = 6\cdot74\ \dot{5}\dot{5} \\
\cdot8\dot{6}\dot{2} & = \ \ \cdot86\ \dot{2}\dot{6} \\
\hline
 & \ 5\cdot88\ \dot{2}\dot{9} \quad \textit{Ans.}
\end{array}$$

EXAMPLES. 97.

Perform the operations indicated below.

1. $3\cdot7\dot{6}+\cdot0\dot{2}$.
2. $\cdot7\dot{8}\dot{9}+\cdot00\dot{3}$.
3. $1\cdot0\dot{4}+2\cdot0\dot{3}+8\cdot01\dot{7}$.
4. $3\cdot07\dot{2}+3\cdot4+\cdot0\dot{1}2\dot{3}$.
5. $3\cdot45+\cdot\dot{6}+\cdot7\dot{1}\dot{2}$.
6. $\cdot0312+\cdot02\dot{3}\dot{1}+\cdot\dot{9}7\dot{6}$.
7. $2\cdot8\dot{2}+\cdot0\dot{3}\dot{4}+\cdot00\dot{1}\dot{4}$.
8. $8\cdot3\dot{1}+\cdot\dot{6}+\cdot00\dot{2}$.
9. $10\cdot0\dot{1}+\cdot000\dot{5}+\cdot\dot{3}$.
10. $7\cdot392+\cdot3\dot{7}+\cdot23\dot{2}$.

11. $\cdot 00\dot{7}+\cdot 0\dot{8}\dot{2}+\cdot 0\dot{1}2\dot{3}$.
12. $1\cdot 12\dot{3}+3\cdot\dot{7}\dot{6}+\cdot\dot{4}57\dot{6}$.
13. $1\cdot 30103+9\cdot\dot{7}+8\cdot 0\dot{9}3\dot{4}$.
14. $\cdot 00\dot{3}+\cdot 0\dot{0}\dot{3}+\cdot\dot{0}0\dot{3}$.
15. $1\cdot\dot{3}+\cdot 0\dot{2}\dot{3}+\cdot\dot{1}23\dot{4}+9\cdot 7$.
16. $\cdot 004+\cdot\dot{3}\dot{7}+\cdot\dot{2}3\dot{4}+1\cdot\dot{1}$.
17. $7\cdot 31\dot{2}347\dot{6}+1\cdot 68\dot{7}652\dot{3}$.
18. $\cdot\dot{7}\dot{4}+3\cdot 00\dot{1}+2\cdot 123\dot{4}$.
19. $72+3\cdot 0\dot{1}2\dot{3}+\cdot 00\dot{1}23\dot{4}$.
20. $1\cdot 3\dot{4}56\dot{3}+2\cdot\dot{6}54\dot{3}$.
21. $3\cdot\dot{1}34\dot{7}+7\cdot 0\dot{3}\dot{2}+\cdot 0\dot{7}+1\cdot 345+\cdot 00\dot{7}\dot{9}$.
22. $1\cdot 3\dot{7}\dot{6}+\cdot 23\dot{7}0\dot{2}+\cdot 0001+\cdot\dot{6}+\cdot\dot{3}\dot{7}$.
23. $4\cdot 0\dot{3}4\dot{5}+7\cdot 234+81+\cdot 0\dot{4}56\dot{7}+\cdot 0\dot{3}+\cdot\dot{1}\dot{2}$.
24. $3\cdot 7\dot{6}-\cdot 00\dot{7}\dot{2}$.
25. $4\cdot 1\dot{3}0\dot{2}-1\cdot 0\dot{5}\dot{2}$.
26. $\cdot\dot{4}32\dot{5}-\cdot 03\dot{7}6\dot{4}$.
27. $2-\cdot\dot{7}\dot{6}-\cdot\dot{3}2\dot{1}$.
28. $3\cdot 46-\cdot 072\dot{3}\dot{4}$.
29. $3\cdot 4\dot{7}6\dot{8}-1\cdot 004$.
30. $7-\cdot 23\dot{4}7\dot{6}$.
31. $\cdot 9-\cdot 00\dot{8}\dot{9}$.
32. $9\cdot 46\dot{8}-3\cdot 123$.
33. $2\cdot\dot{4}67\dot{9}-\cdot 00\dot{3}4\dot{5}$.
34. $1-\cdot\dot{1}0\dot{2}-\cdot 4\dot{6}$.
35. $3\cdot\dot{8}97\dot{2}-\cdot 00\dot{3}\dot{4}$.
36. $\cdot\dot{7}28\dot{4}-\cdot 0\dot{1}2\dot{3}$.
37. $3\cdot\dot{7}\dot{6}-\cdot\dot{1}234\dot{5}$.
38. $\cdot 1\dot{2}34\dot{5}-\cdot 000\dot{3}\dot{7}$.
39. $789\cdot 07\dot{3}\dot{8}-18\cdot 0003\dot{2}5\dot{6}$.
40. $30-\cdot 376\dot{9}803\dot{4}$.

156. Multiplication and Division of Recurring Decimals.

Rule. Reduce the decimals to vulgar fractions; find the product or quotient as a vulgar fraction and reduce it back to the equivalent decimal. *But in the case of division, if the dividend and divisor are both recurring decimals, it will be generally convenient to make them similar before reducing to vulgar fractions.*

Example 1. Multiply $\cdot\dot{0}\dot{9}$ by $7\cdot\dot{3}$.

$$\cdot\dot{0}\dot{9}\times 7\cdot\dot{3}=\frac{9}{99}\times\frac{73-7}{9}=\frac{1}{11}\times\frac{22}{3}=\frac{2}{3}=\cdot\dot{6}. \quad \textit{Ans.}$$

Example 2. Divide $\cdot\dot{6}$ by $\cdot 75$.

$$\cdot\dot{6}\div\cdot 75=\frac{6}{9}\div\frac{75}{100}=\frac{2}{3}\times\frac{4}{3}=\frac{8}{9}=\cdot\dot{8}. \quad \textit{Ans.}$$

Example 3. Divide $\cdot 7\dot{3}\dot{2}$ by $\cdot 02\dot{7}$.

$$\cdot 7\dot{3}\dot{2}\div\cdot 02\dot{7}=\cdot 73\dot{2}\dot{3}\div\cdot 02\dot{7}\dot{7}=\frac{7250}{9900}\div\frac{275}{9900}$$
$$=\frac{7250}{275}=\frac{290}{11}=26\cdot\dot{3}\dot{6}. \quad \textit{Ans.}$$

EXAMPLES. 98.

Find the value of

1. $\cdot 0\dot{3}\times\cdot 0\dot{6}$.
2. $4\cdot\dot{8}\times\cdot\dot{2}\dot{4}$.
3. $\cdot\dot{2}\dot{7}\times 4\cdot\dot{9}\dot{0}$.
4. $\cdot\dot{1}\dot{2}\times 1\cdot\dot{3}$.
5. $2\cdot\dot{4}\times\cdot 0\dot{4}$.
6. $7\cdot\dot{6}\times 6\cdot\dot{7}$.
7. $\cdot\dot{3}\div\cdot\dot{6}$.
8. $\cdot 3\dot{4}\div\cdot 00\dot{3}\dot{2}$.
9. $8\cdot 0\dot{2}\div\cdot 00\dot{3}\dot{4}$.
10. $\cdot 3\dot{4}5\dot{6}\div\cdot 22\dot{7}\dot{6}$.
11. $3\cdot 9\dot{2}\div 1\cdot 40\dot{3}$.
12. $\cdot\dot{1}4285\dot{7}\div\cdot\dot{1}\dot{8}$.
13. $\cdot 0\dot{8}\dot{1}\div\cdot\dot{3}4\dot{6}$.
14. $\cdot 0\dot{2}3\dot{4}\div\cdot\dot{2}\dot{8}$.
15. $\cdot 3\dot{1}2\dot{3}\div\cdot\dot{0}04\dot{5}$.

157. Complex Fractions involving Decimals.

Example. Simplify $\dfrac{\cdot\dot{3}\text{ of }\frac{5}{6}}{\cdot 5\times\cdot\dot{1}}+\dfrac{\cdot 3\dot{5}}{\cdot 0\dot{8}}$.

$$\frac{\cdot\dot{3}\text{ of }\frac{5}{6}}{\cdot 5\times\cdot\dot{1}}+\frac{\cdot 3\dot{5}}{\cdot 0\dot{8}}=\frac{\frac{1}{3}\times\frac{5}{6}}{\frac{1}{2}\times\frac{1}{9}}+\frac{\frac{32}{90}}{\frac{8}{90}}=\frac{1}{\cancel{3}}\times\frac{5}{\underset{\cancel{3}}{\cancel{6}}}\times\frac{\cancel{2}}{1}\times\frac{\cancel{9}}{1}+\frac{\overset{4}{\cancel{32}}}{\cancel{90}}\times\frac{\cancel{90}}{\cancel{8}}$$

$$=5+4=9. \quad \textit{Ans.}$$

EXAMPLES. 99.

Simplify, giving each answer in decimals,

1. $\dfrac{\cdot 0075+2\cdot 1}{\cdot 0175}$.　　2. $\dfrac{4\cdot 255+\cdot 0064}{\cdot 00032}$.　　3. $\dfrac{\cdot 00\dot{1}\times\cdot 05}{\cdot 0022}$.

4. $\dfrac{6\cdot 27\times 0\cdot 5}{(\frac{1}{2}\text{ of }\frac{3}{4})\times 8\cdot 36}\div\dfrac{(\frac{1}{4}\text{ of }\frac{1}{10})\times(\cdot 75\text{ of }21\cdot\dot{3})}{(\frac{2}{3}\text{ of }\frac{5}{6})+1\cdot\dot{4}}$.

5. $\dfrac{4\cdot\dot{2}-3\cdot 1\dot{4}}{1\cdot 3+2\cdot\dot{1}0\dot{2}}$ of $\dfrac{1\cdot 3\text{ of }4}{\cdot 37\text{ of }8\cdot\dot{8}\dot{1}}$.　　6. $\dfrac{1\cdot 8\dot{3}+2\cdot 041\dot{6}+\cdot\dot{3}-3\frac{3}{8}}{1\cdot 0025+\cdot 0625-1\frac{1}{16}}$.

7. $\dfrac{\cdot 12\text{ of }(\cdot 0104-\cdot 002)+\cdot 36\times\cdot 002}{\cdot 12\times\cdot 12}$.

8. $\dfrac{3\cdot 125}{2\cdot 16}$ of $\dfrac{\cdot 2\dot{4}}{\cdot 125}\div\dfrac{2\cdot\dot{2}}{1\cdot 5}$ of $\dfrac{187\cdot 5}{3\cdot 42}$.

9. $\left\{37+\dfrac{3\cdot 7\dot{0}3\dot{7}}{100}\right\}\times\cdot 54$.　　10. $\dfrac{\frac{5}{6}\text{ of }\frac{18}{25}+\frac{1}{7}\times 2\cdot\dot{3}}{3-(\frac{8}{5}+\frac{7}{10})\div 2\cdot\dot{3}\dot{6}}$.

11. $\dfrac{\cdot 1\times\cdot 1\times\cdot 1+\cdot 01\times\cdot 01\times\cdot 01}{\cdot 2\times\cdot 2\times\cdot 2+\cdot 02\times\cdot 02\times\cdot 02}$.　　12. $\dfrac{\cdot 044\times 2\cdot 1}{\cdot 000035}\div\dfrac{3\cdot\dot{0}7692\dot{3}}{2\cdot\dot{3}\times 5\cdot\dot{6}}$.

13. $\dfrac{2\cdot 8\text{ of }2\cdot\dot{2}\dot{7}}{1\cdot\dot{3}\dot{6}}+\left\{\dfrac{4\cdot\dot{4}-2\cdot 8\dot{3}}{1\cdot\dot{3}+2\cdot 62\dot{9}}\text{ of }8\cdot 2\right\}$.

14. $\dfrac{\cdot 175-\cdot 116\text{ of }\dfrac{1\frac{1}{3}}{3\frac{1}{2}}}{\cdot 083\text{ of }\dfrac{1\frac{5}{7}}{2\frac{6}{7}}+\cdot 55}$.　　15. $\dfrac{\cdot\dot{0}7692\dot{3}}{\cdot\dot{0}3\dot{7}}\times\dfrac{999}{\cdot\dot{0}2\dot{7}}\times\dfrac{\cdot 001}{111}\times\dfrac{13}{\cdot\dot{0}0\dot{9}}$.

16. $\dfrac{9\cdot\dot{6}}{3\cdot\dot{3}}\times\dfrac{14\cdot 02\dot{3}}{\cdot 9}\times 1\dfrac{1}{29}\times\cdot\dot{3}\times 1\cdot\dot{7}4\dot{1}\div\cdot\dot{0}0\dot{6}\times\dfrac{30}{4207}$.

XXVII. DECIMAL MEASURES.

158. *Example* 1. Reduce ₨3·4 to pies.

Process :

$$\begin{array}{r} ₨3{\cdot}4 \\ 16 \\ \hline 54{\cdot}4a. \\ 12 \\ \hline 652{\cdot}8p. \end{array} \quad \textit{Ans.}$$

Example 2. Find the value of 4·135 of £1.

Process :

$$\begin{array}{rr} & £4{\cdot}135 \\ & 20 \\ \hline s. & 2{\cdot}70\cancel{0} \\ & 12 \\ \hline d. & 8{\cdot}4 \end{array}$$

The £4 is not reduced to shillings.

The 2*s*. is not reduced to pence.

∴ 4·135 of £1 = £4. 2*s*. 8·4*d*.

Example 3. How many rupees, annas and pies are there in ·522 of ₨5 ?

Process :

$$\begin{array}{rr} & {\cdot}522 \\ & 5 \\ \hline ₨ & 2{\cdot}61\cancel{0} \\ & 16 \\ \hline a. & 9{\cdot}76 \\ & 12 \\ \hline p. & 9{\cdot}12 \end{array}$$

∴ ·522 of ₨5 = ₨2. 9*a*. 9·12*p*.

Example 4. Find the value of ·25 of £9. 7*s*. 6*d*.

Process : £9. 7*s*. 6*d*. = 2250*d*.

$$\begin{array}{r} {\cdot}25 \\ 2250 \\ \hline 125 \\ 50 \\ 50 \\ \hline 12\,)\,562{\cdot}5\cancel{0}d. \\ \hline 20\,)\,46s.\ 10{\cdot}5d. \\ \hline £2.\ 6s.\ 10{\cdot}5d. \end{array}$$

∴ ·25 of £9. 7*s*. 6*d* = £2. 6*s*. $10\frac{1}{2}$*d*.

Example 5. Find the value of $\cdot2\dot{3}$ of ₨10. 5*a*.

Process : $\cdot2\dot{3}$ of ₨10. 5*a*. = $\frac{7}{30}$ of ₨10. 5*a*. = etc.

EXAMPLES. 100.

Reduce

1. R7·15 to pies.
2. ·0234375 of R1 to pies.
3. £·134375 to pence.
4. ·00375 of £1 to farthings.
5. ·03125 of R5 to pies.
6. ·045 of £7 to farthings.
7. R8·2$\dot{3}$ to pies.
8. ·0$\dot{7}$ of £5 to pence.
9. ·895 cwt. to ounces.
10. 3·985 poles to inches.

Express as compound quantities :

11. R7·325.
12. £3·35.
13. R2·02.
14. 2·575 of 15*a.*
15. 3·45 of 16*s.*
16. ·06 of R13·5.
17. 3·725 of R9·2.
18. ·032 of 12 yd.
19. ·234 ton.

Find the value of

20. ·625 of R1. 4*a.* 4*p.*
21. ·725 of R9 6*a.*
22. R9. 2*a.* × 1·35.
23. ·6 of R7. 9*a.* 10*p.*
24. 3·9 of R11. 9*a.*
25. ·079 of R35·5.
26. ·256 of £3. 4*s.* 9*d.*
27. ·1875 of 9*s.* $4\frac{1}{2}$*d.*
28. ·0625 of 3·6*s.*
29. R3. 3*a.* 8*p.* × ·785.
30. £6 × ·78125.
31. 3*s.* $6\frac{1}{2}$*d.* × ·45.
32. 3 md. 7 seers 9 ch. × 3·24.
33. 2 tons 3 cwt. 2 qr. 8 lb. × ·65.
34. 3 po. 2 yd. $1\frac{1}{2}$ in. × ·725.
35. 1 da. 3 hr. 3 min. 7 sec. × ·825.
36. 3·$\dot{4}$ of R2. 4*a.*
37. ·$\dot{6}\dot{3}$ of 3*s.* $6\frac{1}{3}$*d.*
38. R7. 9*a.* ÷ ·0$\dot{6}$.
39. R3. 4*a.* 9*p.* ÷ ·422.
40. £7. 8*s.* 2*d.* ÷ ·044.
41. 11·1375 of R6. 8*a.* − ·56 of R7. 8*a.*
42. ·8$\dot{3}$ of R2. 8*a.* + ·6 of R4. 11*a.* + 2·05 of R5.
43. ·375 of R9 + ·8$\dot{3}$ of 10*a.* − ·$\dot{6}$ of 6*p.*
44. ·01$\dot{6}$ of R260. 2*a.* 6*p.* + ·$\dot{3}5\dot{1}$ of R13. 14*a.* + 1·0$\dot{0}$03$\dot{3}$ of R7. 14*a.* 3*p.*
45. ·03125 of R2 + ·$\dot{7}2\dot{9}$ of R$3\frac{1}{12}$ + ·$\dot{7}2\dot{9}$ of R$3\frac{5}{8}$.
46. £·634375 + ·025 of 25*s.* + ·325 of 30*s.*
47. 8·71875 of 8*d.* + 1·146875 of 6*s.* 8*d.* − ·0525 of 1 guinea.
48. 6·8$\dot{3}$ of £3·867708$\dot{3}$ + 5·8 of £2·411458$\dot{3}$ − 4·375 of £1·3.

Arrange in order of magnitude :

49. $\frac{11}{12}$ of R3. 9*a.*, ·025 of R100. 10*a.*, ·3$\dot{2}$ of R5. 8*a.*
50. ·003$\dot{1}$ of £1, ·256 of 1*s.*, $3\frac{1}{3}$ of 1*d.*
51. What is the sum, ·75 of which is R3. 9*a.* 2*p.* ?
52. $\frac{3}{4}$ of ·72 of a sum of money is 3*s.* 6*d.* ; what is ·0$\dot{3}$ of the sum ?
53. Simplify $\dfrac{\cdot625 \text{ of } £143.\ 12s. + \cdot625 \text{ of } £71.\ 16s.}{\frac{5}{8} \text{ of } 5175}$.

54. Simplify $\cdot 42\dot{6}$ of $\frac{3\cdot\dot{3}}{\cdot 08}$ of $\frac{3}{\cdot\dot{7}3\dot{5}}$ of $\frac{\cdot 147\times 4\cdot\dot{4}}{11\cdot 1}$ of £1. 17*s*. 6*d*.

55. Multiply ·892 of ₹16. 5*a*. 4*p*. by 4·678.

56. Find the value of $\cdot\dot{8}5714\dot{2}$ of 2·0625 tons + $\cdot\dot{5}7142\dot{8}$ of 3·375 cwt. + $\cdot\dot{7}1428\dot{5}$ of 1·25 qr. + $\cdot\dot{2}8571\dot{4}$ of 10·5 lb.

57. Find the value of $\cdot\dot{0}\dot{9}$ of 1·5 md. + $\cdot\dot{2}\dot{7}$ of 2·25 md. + $\dot{6}\dot{3}$ of 7·75 md. + $\cdot\dot{4}\dot{5}$ of ·7 md.

58. Find the greatest sum of money which is contained in each of ·25 of 5*s*. 6*d*. and $\cdot 0\dot{5}$ of £1 a whole number of times.

159. The following examples illustrate the *converse* operation:

Example 1. Reduce 1000 pies to rupees.

$$1000p. = ₹\frac{1000}{12\times 16} = ₹\frac{125}{24} = ₹5\cdot 208\dot{3}. \quad \textit{Ans.}$$

Example 2. Reduce £1. 3*s*. 6*d*. to the decimal of £1.

$$£1.\ 3s.\ 6d. = £1.\ 42d. = £1\frac{42}{12\times 20} = £1\frac{7}{40} = £1\cdot 175;$$

∴ the decimal = 1·175.

Example 3. Express $\cdot\dot{3}$ of ₹1. 3*a*. 6*p*. as the decimal of 4*a*. 10*p*.

$$\text{The decimal} = \frac{\cdot\dot{3}\text{ of }₹1 . 3 . 6}{4a.\ 10p.} = \frac{\frac{1}{3}\times 234}{58} = \frac{234}{3\times 58} = \frac{39}{29} = 1\cdot 3448\ldots$$

EXAMPLES. 101.

Reduce

1. 3333 pies to rupees.

2. 8446*q*. to pounds.

3. 10000 lb. to tons.

4. 90000 in. to miles.

5. 66666 sec. to days.

6. 39 guineas to pounds.

Express each of the following as a decimal of its *highest* denomination.

7. 7*a*. 9*p*. **8.** ₹3. 10*a*. 3*p*. **9.** ₹5. 5*a*. 5*p*.

10. 8*s*. 6*d*. **11.** £1. 3*s*. 8*d*. **12.** £7. 6*s*. $4\frac{1}{2}$*d*.

13. 1 md. 15 seers. **14.** 3 cwt. $3\frac{1}{3}$ qr. **15.** 5 po. 4 yd.

16. 7 da. $5\frac{3}{4}$ hr. **17.** 1 ac. 20 yd. 3 ft. **18.** 7°. 2′. 20″.

In the following examples, reduce the first of the two given quantities to the decimal of the second.

19. ₹3. 4*a*. 9*p*.; ₹5.

20. £7. 10*s*. $4\frac{1}{2}$*d*.; £10.

21. $9a.\ 4p.$; $11a.\ 3p.$
22. R7. $9a.\ 10p.$; R12. $4a.\ 4p.$
23. $7s.\ 6d.$; $15s.\ 7d.$
24. £3. $10s.\ 9\frac{1}{2}d.$; £6. $2s.\ 4\frac{1}{2}d.$
25. $\frac{3}{4}$ of £1. $8s.\ 6d.$; £1.
26. $\frac{7}{8}$ of R3. $9a.\ 4p.$; R3.
27. ·375 of R10. $10a.\ 10p.$; R3. $13a.\ 3p.$
28. $9a.\ 8p.$; $\cdot 3\dot{8}$ of R3. $4a.$
29. ·35 of £7. $3s.\ 4\frac{1}{2}d.$; $\cdot 0\dot{5}$ of £3.
30. $\cdot 00\dot{3}$ of £1. ; $\cdot\dot{7}$ of $9s.\ 4\frac{1}{2}d.$
31. ·25 of $3a.\ 4p.$; $\cdot 0\dot{6}$ of R3.
32. $2\frac{2}{25}$ of £2. $6s.\ 5\frac{3}{4}d.$; £18. $17s.\ 10\frac{3}{4}d.$

33. Express $\frac{5}{8}$ of $12s.\ 6d.$ + ·625 of $7s.\ 6d.$ − $\cdot\dot{5}0\dot{5}$ of $16s.\ 6d.$ as the decimal of £1.

34. Reduce $\frac{3}{8}$ of R·05 + $\frac{7}{10}$ of $4a.$ + $\frac{3}{8}$ of R1 to the decimal of R$\frac{2}{25}$.

35. Express $\cdot\dot{4}2857\dot{1}$ of £1·05 + $\cdot 3\dot{8}$ of 1·5$s.$ as the decimal of £43. $2s.\ 6d.$

36. Reduce $\cdot\dot{2}4\dot{6}$ of $9s.\ 3d.$ + $\cdot 25\dot{9}$ of £1. $5s.$ + $\cdot 0\dot{2}$ of £3. $7s.\ 6d.$ to the decimal of $\cdot 0\dot{3}$ of £90.

37. Reduce ·062435 of £100 + 7·4375 of $10s.$ + $1\cdot 35\dot{6}$ of $7s.\ 6d.$ + 2·784 of $2\frac{1}{2}d.$ to the decimal of £29. $10s.\ 7\frac{1}{2}d.$

38. What decimal of R3. $9a.$ must be added to $\cdot 0\dot{7}\dot{5}$ of $5a.\ 6p.$ to make the sum equal to 1 anna ?

39. What decimal of £6. $10s.$ must be taken from $\frac{3}{4}$ of £9 that the remainder may be £6. $10s.$?

40. Express £874. $13s.\ 4d.$ × 3·75 as the decimal of £10000.

MISCELLANEOUS EXAMPLES. 102.

1. Give the local value of each of the significant digits in ·02073.

2. Express the difference between $2\cdot\dot{7}\dot{6}$ and 2·76, (i) by a circulating decimal, and (ii) by a vulgar fraction.

3. Express $\frac{1}{3}(3\frac{1}{2}+2\frac{2}{3}-4)$ as a decimal, and $\cdot 6+\frac{2}{11}$ of ·025 + $3\cdot 0\dot{6}$ as a vulgar fraction.

4. Reduce $\frac{3}{25}$ of 2·35 ÷ 1000 to a decimal.

5. Find the least number which must be subtracted from the sum of 2·36 and 3·002 that the remainder may be an integer.

6. Find the price of 321 yards of cloth at 11·25 annas per yard.

7. Find the total weight of 324 bags, each 13·75 lb.

8. By what decimal do we divide $3\frac{4}{5}$, if the quotient is 7·5 ?

9. R720 is ·08 of what amount ?

10. If the divisor be 2·36 and the quotient ·125 of the divisor, what must the dividend be ?

11. Divide 64·09 by 49·3, and arrange the divisor, dividend and quotient in order of magnitude.

12. If the diameter of a pice be 1·025 inches, how many must be placed in contact along a straight line to extend from Calcutta to Hugly, a distance of 24·6 miles?

13. How often will a wheel, 2·75 yards in circumference, turn in a distance of 12·5 miles?

14. A vessel holds 3·256 gallons; how many times can it be filled from a cask of 96 gallons? Will there be any remainder?

15. How many times can you subtract 3·01 from 65·23, and what is the remainder?

16. Express as a decimal the continued product of $\frac{3}{8}$, $\frac{2\frac{1}{2}+1\cdot5}{8\cdot75}$ and $\frac{2\frac{11}{12}}{\cdot\dot{9}}$.

17. Express 21·43 crowns + 18·52 shillings in pence.

18. Subtract 4·42 cwt. from 7·28 tons.

19. Express 2·75 oz. + ·075 cwt. in pounds.

20. Find the rent of 32·25 acres at £1·025 per acre.

21. If the product of ·064 and a certain number be divided by ·00008, the quotient is 3404; find the number.

22. A book containing 219 leaves is 1·34 inches thick; allowing ·06 of an inch for the cover, find to 5 decimal places the thickness of the paper.

23. A roller 4·03 ft. in circumference makes 34·04 revolutions in passing from one end of a lawn to another: what is the length of the lawn?

24. From a rod 2 yards long, portions each ·063 of an inch in length are cut off; how many such portions can be cut off, and what will be the length of the remaining piece?

25. Find a decimal which shall differ from $\frac{17}{21}$ by less than $\frac{1}{10000}$

26. Multiply 9·035 by itself in two lines.

27. Multiply 37·056 by 12·10411 in three lines.

28. Find the least number of articles, costing R2·375 each, that can be purchased for an integral number of rupees.

29. Find the smallest number of articles, costing £2. 6*s*. 2·37*d* each, that you can buy for an exact number of pounds.

30. *A* did ·025 of a piece of work, and *B* ·825; how much was left to be done?

31. A boy, after giving away ·8 of his pocket-money to one companion, and ·06 of the remainder to another, has 7*a*. 10*p*. left; how much had he at first?

32. A man received $\cdot 3\dot{8}$ of $\cdot 0\dot{3}$ of a property, and sold $\cdot\dot{3}$ of his own share for R350; what would be the value of the whole property at the same rate?

33. A gallon contains 277·274 cubic inches; how many cubic yards are there in 200 bushels?

34. A cubic foot of water weighs 62·35 lb. avoir.; what would be the error in calculating the weight of 30 cubic feet on the approximate supposition that a cubic foot of water weighs 1000 oz.?

35. A is ·75 times as old as B, and C $\cdot 7\dot{5}$ times as old as B; A is 15 years old: how old is C?

36. Four bells toll at intervals of 1·3, 1·4, 1·5 and 1·6 seconds, beginning together; after what interval will they toll together again?

37. Find the largest sum of money which is contained in £3·75 and £2·125 a whole number of times.

38. Divide R50 into two parts such that one part may be $\cdot\dot{6}$ of the other.

39. Divide £52 between A, B, C in such a manner that B may receive $\cdot\dot{3}$ of A, and C $\cdot\dot{3}$ of B.

40. Express $\dfrac{8\frac{11}{63}}{\frac{7}{8} \text{ of } 2\cdot 25}$ of $\dfrac{\cdot 1625}{\frac{1}{16} \text{ of } 5\frac{1}{7}} \div \left(\dfrac{2}{21}+\dfrac{7}{81}\right)$ as a fraction of $\left\{37+\dfrac{3\cdot 7\dot{0}3\dot{7}}{100}\right\}$ of ·54.

XXVIII. APPROXIMATION.

160. It is often inconvenient, and not always possible, to find an *exact* decimal equivalent to a proposed number. In such cases we may proceed to a few places of decimals and indicate by dots (...) that the work has not terminated. Thus $\frac{22}{23}$ = ·95652... If however we wish to *approximate* to the result by terminating our work at any specified place, we should increase the last digit retained by 1 if the first digit rejected be 5 or greater than 5. Thus $\frac{22}{23}$ = ·957 *correct to three places of decimals* or *to the nearest thousandth;* also $\frac{22}{23}$ = ·9565 *to four places.*

Note. It will be easily seen that the difference of ·957 and ·95652... is less than the difference of ·95652... and ·956; hence ·957 represents ·95652... more accurately than ·956. It may be noticed that the approximate result is less than the actual result when the first figure rejected is less than 5, but greater when not less.

161. Contracted Addition and Subtraction.

Example 1. Find the sum of $\cdot 23\dot{6}\dot{7}$, $\cdot 3\dot{1}7\dot{8}$ and $1\cdot 62$ correct to four places of decimals.

We write down each decimal to 7 places, and obtain correctly 5 decimal places in the sum; the required result is then obtained by rejecting the fifth place.

$$\begin{array}{r|l} \cdot 2367 & 676 \\ \cdot 3178 & 178 \\ 1\cdot 62 & \\ \hline 2\cdot 1745 & 8\ldots \end{array} = 2\cdot 1746. \quad \textit{Ans.}$$

Example 2. Find the difference between $\cdot 6\dot{3}2\dot{1}$ and $\cdot 00\dot{8}$ correct to five places of decimals.

Process:

$$\begin{array}{r|l} \cdot 63213 & 213 \\ \cdot 00888 & 888 \\ \hline \cdot 62324 & 3\ldots \end{array} = \cdot 62324. \quad \textit{Ans.}$$

Example 3. Find the value of $1+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\ldots$ correct to 3 places of decimals.

$$\begin{array}{lrcccl}
 & & & & 1 = & 1\cdot 000'000 \\
\therefore & \frac{1}{1.2} & = & \frac{1}{2} & = & \cdot 500\,000 \\
\therefore & \frac{1}{1.2.3} & = & \frac{\cdot 5}{3} & = & \cdot 166\,666 \\
\therefore & \frac{1}{1.2.3.4} & = & \frac{\cdot 166666}{4} & = & \cdot 041\,666 \\
\therefore & \frac{1}{1.2.3.4.5} & = & \frac{\cdot 041666}{5} & = & \cdot 008\,333 \\
\therefore & \frac{1}{1.2.3.4.5.6} & = & \frac{\cdot 008333}{6} & = & \cdot 001\,388 \\
\therefore & \frac{1}{1.2.3.4.5.6.7} & = & \frac{\cdot 001388}{7} & = & \cdot 000\,198 \\
\therefore & \frac{1}{1.2.3.4.5.6.7.8} & = & \frac{\cdot 000198}{8} & = & \cdot 000\,024 \\
\therefore & \frac{1}{1.2.3.4.5.6.7.8.9} & = & \frac{\cdot 000024}{9} & = & \cdot 000\,002 \\
\hline
 & \text{and } \therefore \text{ the expression} & & & = & 1\cdot 718,2\ldots \\
 & & & & = & 1\cdot 718, \text{ to 3 places.}
\end{array}$$

Here we stop at $\frac{1}{1.2.3.4.5.6.7.8.9}$, as in the decimals equivalent to the succeeding fractions, the first six figures will be zeros.

EXAMPLES. 103.

1. Find the quotient of 40 divided by 19 correct to four places of decimals.

2. Obtain the decimal equivalent to $\frac{1}{17}$ correct to five places of decimals.

3. Find the value of $\cdot 0312 + \cdot 02\dot{3}\dot{1} + \cdot\dot{9}7\dot{6}$ correct to four places of decimals.

4. Find the sum of 72, $3\cdot 0\dot{1}2\dot{3}$ and $\cdot 00\dot{1}23\dot{4}$ correct to three places of decimals.

5. Find the difference between $\cdot\dot{4}32\dot{5}$ and $\cdot 0\dot{3}76\dot{4}$ correct to four places of decimals.

Find the value, correct to 2 places of decimals, of

6. $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\dots$ **7.** $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$

8. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots$ **9.** $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\dots$

Find the value, correct to 3 places of decimals, of

10. $1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\dots$ **11.** $1+\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\dots$

Find the value, correct to 5 places of decimals, of

12. $\cdot 25+(\cdot 25)^3+(\cdot 25)^5+\dots$ **13.** $1+\frac{1}{1.3}+\frac{1}{1.3.5}+\frac{1}{1.3.5.7}+\dots$

14. $\frac{1}{1}\cdot\frac{1}{2^2}+\frac{1}{2}\cdot\frac{1}{2^4}+\frac{1}{3}\cdot\frac{1}{2^6}+\frac{1}{4}\cdot\frac{1}{2^8}+\dots$

[First express as decimals $\frac{1}{2^2}, \frac{1}{2^4}, \frac{1}{2^6}, \dots\dots$, then divide the results respectively by 1, 2, 3,..., and add.]

15. $\frac{1}{1}\cdot\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}+\frac{1}{7}\cdot\frac{1}{5^7}+\frac{1}{9}\cdot\frac{1}{5^9}+\dots$

CONTRACTED MULTIPLICATION.

162. The following rule will shorten the process of multiplication when the product is required only to a certain number of decimal places.

To multiply two decimals together, retaining, say, 5 decimal places :—"Reverse the multiplier, strike out the decimal points, and place the multiplier under the multiplicand, so that what was its units' figure shall fall under the 5th decimal place of the multiplicand, placing ciphers, if necessary, so that every place of the multiplier shall have a figure above it. Proceed to multiply as usual, beginning each figure of the multiplier with the one which is in the place

to its right in the multiplicand : do not set down from this product but carry its *nearest ten** to the next, and proceed. Place the first figures of all the lines under one another ; add as usual ; and mark off 5 places from the right for decimals."—[De Morgan.]

Example. Multiply 7·2078 by 2·3072, retaining 5 places ; ·00705328 by 12·30523, retaining 6 places ; and 29·82 by ·00727, retaining 4 places of decimals.

(i)	(ii)	(iii)
720780	705328	29820
27032	3250321	72700
1441560	70533	2087
216234	14106	60
5045	2116	20
144	35	·2167
16·62983	1	
	·086791	

Note. The last figure in the product thus obtained may not be always correct, and to ensure its accuracy we must carry the process one place farther than is required to be retained.

CONTRACTED DIVISION.

162a. The following rule will shorten the process in division when the quotient is required to be correct only to a certain number of decimal places.

Make the divisor a whole number ; and determine by inspection (or by taking one step in the ordinary way), how many figures there will be in the integral part of the quotient. In the divisor retain (from the left) as many figures as there are to be in the whole quotient—integral part as well as decimal ; and strike off the rest. Proceed one step with this new divisor, but to the product of its first figure by the quotient-figure, carry the *nearest ten* from the preceding figure. Instead of bringing down a figure to the remainder, strike off another figure from the divisor, and proceed as before, until no figure is left in the divisor.

If the number of figures in the divisor be less than the number of quotient-figures to be obtained, proceed in the ordinary way until the number of quotient-figures still to be obtained, is one less than the number of figures in the divisor. As soon as this happens, instead of bringing down a figure to the remainder, strike off a figure from the end of the divisor, and then proceed as in the preceding case.

* That is, carry 1 if the product is a number from 5 to 14 ; carry 2 if it is from 15 to 24 ; carry 3 if it is from 25 to 34 ; etc. ; if the product is 4 or less, we ignore it.

Example. Divide 29·431542 by 3·25348 to 3 decimal places; and 673·1489 by ·41432 to 2 places.

```
(i) 3.2.5.3.4.8 ) 2943154·2 ( 9·046
                  29281
                    150
                    130
                     20
                     19
                      1

(ii) 4.1.4.3.2 ) 6731489 0· ( 1624·70
                 41432
                 258828
                 248592
                  10236
                   8286
                   1950
                   1657
                    293
                    290
                      3
```

EXAMPLES. 103a.

Multiply

1. 21·1324 by ·345721 to 3 decimal places.
2. ·32504 by 13·0254 to 3 ... ...
3. ·453 by ·01694 to 4 ... ...
4. 375·76843 by 3·14159 to 4 ... ...
5. 71·032751 by 2·6719238 to 5 ... ...
6. 65·00763 by ·9876 to 5 ... ...
7. ·03281674 by 234·781 to 6 ... ...
8. ·0008127 by 483·2716 to 6 ... ...
9. $4 \cdot 68\dot{3}$ by $14 \cdot \dot{2}9\dot{3}$ to 3 ... ...
10. $1 \cdot 8\dot{2}35\dot{7}$ by $\cdot 0\dot{7}8\dot{5}$ to 6 ... ...

Divide

11. 76·2307 by 47·12345 to 3 decimal places.
12. 3·3706 by 9·7846 to 3 ... ...
13. 32·791 by 26·67. to 3 ... ...
14. 378·325 by 30·732 to 3 ... ...
15. 36·7802 by 312·32 to 4 ... ...
16. 728·389 by 3·76 to 4 ... ...
17. 3892·762 by 7·343 to 5 ... ...
18. 23·78934 by ·00289 to 5 ... ...
19. 13·2346891 by ·01234031 to 6 ... ...
20. 132·405678 by ·000122134 to 7 ... ...
21. $3 \cdot 7\dot{2}\dot{5}$ by $13 \cdot \dot{2}3\dot{4}$ to 3 ... ...
22. $1 \cdot \dot{8}235\dot{7}$ by $\cdot 0\dot{7}8\dot{5}$. to 6 ... ...

XXIX. PRACTICE.

163. **An aliquot part** of a quantity is a quantity which can be expressed as a fraction of that quantity, having *unity* for its numerator.

Thus 4*a*., being $\frac{1}{4}$ of R1, is an aliquot part of R1; 2*s*. 6*d*., which is $\frac{1}{8}$ of £1, is an aliquot part of £1.

164. **Simple Practice** is a convenient method of finding, by means of aliquot parts, the cost of a *simple quantity*, when the cost is given of the unit-quantity, in terms of which the simple quantity is expressed.

Example. Find the value of 32 cwt. of wheat at R3. 8*a*. per cwt.

Compound Practice is a convenient method of finding, by means of aliquot parts, the cost of a *compound quantity*, when the cost is given of one of the units, in terms of which the compound quantity is expressed.

Example. Find the value of 7 cwt. 3 qr. of wheat at R3. 8*a*. per cwt.

SIMPLE PRACTICE.

165. The following examples will explain the method of Simple Practice.

Example 1. Find the price of 23 md. of rice at R3. 13*a*. 9*p*. per md.

	R.	*a*.	*p*.	
	23	0	0	=price at R1 per md.
			3	
	69	0	0	=price at R3 per md.
8*a*.=$\frac{1}{2}$ of R1	11	8	0	= " " 8*a*. " "
4*a*.=$\frac{1}{2}$ of 8*a*.	5	12	0	= " " 4*a*. " "
1*a*.=$\frac{1}{4}$ of 4*a*.	1	7	0	= " " 1*a*. " "
6*p*.=$\frac{1}{2}$ of 1*a*.		11	6	= " " 6*p*. " "
3*p*.=$\frac{1}{2}$ of 6*p*.		5	9	= " " 3*p*. " "
	R88	12	3	=price at R3. 13*a*. 9*p*. per md.

Note 1. Since R3.13.9 is the difference between R4 and 2*a*. 3*p*., a shorter method would be to find the price at 2*a*. 3*p* per md. and subtract it from the price at R4 per md.

Thus ;

	R.	a.	p.	
	23	0	0	
			4	
	92	0	0	=price at R4 per md.
	3	3	9	= ,, ,, 2*a*. 3*p*. ,, ,,
	R88	12	3	=price at R3. 13*a*. 9*p*. per md.

	R.	a.	p.	
	23	0	0	
2*a*.=$\frac{1}{8}$ of R1.	2	14	0	
3*p*.=$\frac{1}{8}$ of 2*a*.		5	9	
	R3	3	9	=price at 2*a*. 3*p*. per md.

Example 2. Find the cost of 9 articles at £10. 12*s*. 6*d*. each.

		£.	s.	d.	
		9	0	0	=cost at £1 each.
				10	
		90	0	0	=cost at £10 each.
10*s*.	$\frac{1}{2}$ of £1.	4	10	0	= ,, ,, 10*s*. ,,
2*s*.	$\frac{1}{5}$ of 10*s*.		18	0	= ,, ,, 2*s*. ,,
6*d*.	$\frac{1}{4}$ of 2*s*.		4	6	= ,, ,, 6*d*. ,,
		£95	12	6	=cost at £10. 12*s*. 6*d*. each.

Note 2. Shorter thus : 10*s*.=$\frac{1}{2}$ of £1 ; 2*s*. 6*d*.=$\frac{1}{4}$ of 10*s*.

Example 3. Find the value of 13$\frac{1}{2}$ cwt. at R7. 10*a*. 3*p*. per cwt.

		R.	a.	p.	
		13	8	0	=value at R1 per cwt.
				7	
		94	8	0	=value at R7 per cwt.
8*a*.	$\frac{1}{2}$ of R1.	6	12	0	= ,, ,, 8*a*. ,, ,,
2*a*.	$\frac{1}{4}$ of 8*a*.	1	11	0	= ,, ,, 2*a*. ,, ,,
3*p*.	$\frac{1}{8}$ of 2*a*.		3	4$\frac{1}{2}$	= ,, ,, 3*p*. ,, ,,
		R103	2	4$\frac{1}{2}$	=value at R7. 10*a*. 3*p*. per cwt.

Or thus :

		R.
		13·5
		7
		94·5
8*a*.	$\frac{1}{2}$ of R1.	6·75
2*a*.	$\frac{1}{4}$ of 8*a*.	1·6875
3*p*.	$\frac{1}{8}$ of 2*a*.	·2109375
		R103·1484375

R.	·1484375
	16
a.	2·3750000
	12
p.	4·500

Ans. R103. 2*a*. 4$\frac{1}{2}$*p*.

Example 4. Find the value of $42\frac{2}{3}$ things at 16*s*. $2\frac{3}{4}$*d*. each.

		£.	s.	d.	
		42	13	4	= value at £1 each.
10*s*.	$\frac{1}{2}$ of £1.	21	6	8	= value at 10*s*. each.
5*s*.	$\frac{1}{2}$ of 10*s*.	10	13	4	= „ „ 5*s*. „
1*s*.	$\frac{1}{5}$ of 5*s*.	2	2	8	= „ „ 1*s*. „
2*d*.	$\frac{1}{6}$ of 1*s*.		7	$1\frac{1}{3}$	= „ „ 2*d*. „
$\frac{1}{2}$*d*.	$\frac{1}{4}$ of 2*d*.		1	$9\frac{1}{3}$	= „ „ $\frac{1}{2}$*d*. „
$\frac{1}{4}$*d*.	$\frac{1}{2}$ of $\frac{1}{2}$*d*.			$10\frac{2}{3}$	= „ „ $\frac{1}{4}$*d*. „
		£34	12	$5\frac{1}{3}$	= value at 16*s*. $2\frac{3}{4}$*d*. each.

EXAMPLES. 104.

Find, by Practice, the cost of the following articles :

1. 400 at ₨3. 4*a*. each.
2. 375 at £2. 5*s*. each.
3. 789 at 1*a*.
4. 728 at 3*d*.
5. 439 at 3*p*.
6. 399 at £4. 4*s*.
7. 874 at 6*a*.
8. 723 at 15*s*.
9. 939 at ₨2. 11*a*.
10. 275 at 4*d*.
11. 475 at 13*a*. 6*p*.
12. 342 at 2*s*. 6*d*.
13. 500 at 7*a*. 3*p*.
14. 942 at 7*s*. 3*d*.
15. 700 at 10*a*. $4\frac{1}{2}$*p*.
16. 374 at $5\frac{1}{2}$*d*.
17. 321 at ₨2. 5*a*. 3*p*.
18. 230 at £7. 10*s*. 6*d*.
19. 366 at ₨7. 11*a*. 9*p*.
20. 767 at £10. 8*s*. 8*d*.
21. 839 at ₨5. 13*a*. 4*p*.
22. 339 at 14*s*. $10\frac{3}{4}$*d*.
23. 454 at ₨15. 7*a*. $10\frac{1}{2}$*p*.
24. 900 at £50. 11*s*. $9\frac{3}{4}$*d*.
25. 900 at ₨42. 10*a*. $7\frac{1}{2}$*p*.
26. 5013 at £55. 19*s*. $1\frac{1}{2}$*d*.
27. 768 at ₨19. 9*a*. 3 *pice*.
28. 1010 at £11. 11*s*. $11\frac{3}{4}$*d*.
29. 8760 at ₨21. 14*a*. 2 *pice*.
30. 4596 at 12*s*. $0\frac{3}{4}$*d*.
31. 555 at ₨89. 3*a*. $5\frac{1}{2}$*p*.
32. 3111 at £12. 12*s*. $3\frac{3}{4}$*d*.
33. 8001 at ₨80. 8*a*. $8\frac{1}{2}$*p*.
34. 10000 at £7. 17*s*. $11\frac{1}{2}$*d*.
35. $346\frac{1}{2}$ at ₨8. 10*a*. 8*p*.
36. $27\frac{3}{5}$ at £8. 16*s*. $7\frac{3}{4}$*d*.
37. $703\frac{2}{3}$ at ₨29. 13*a*. $4\frac{1}{2}$*p*.
38. $301\frac{1}{4}$ at £2. 15*s*. $7\frac{3}{4}$*d*.
39. $821\frac{3}{4}$ at ₨41. 7*a*. $5\frac{3}{4}$*p*.
40. $442\frac{3}{8}$ at £76. 2*s*. $4\frac{3}{4}$*d*.
41. $600\frac{5}{12}$ at ₨12. 12*a*. 2*p*.
42. $249\frac{7}{10}$ at £20. 2*s*. $8\frac{1}{4}$*d*.
43. 39·5 at ₨1. 13*a*. 4*p*.
44. 84·75 at £2. 15*s*. 9*d*.
45. 101·375 at ₨10. 9*a*. 6*p*.
46. 10·875 at £2. 17*s*. $10\frac{1}{4}$*d*.

Reflecting Telescopes.—In all these, instead of an object-glass consisting of a convex lens, a concave mirror, called the object-mirror, is used to form an image of the object.

In *Herschel's* telescope the aperture is very large, and the

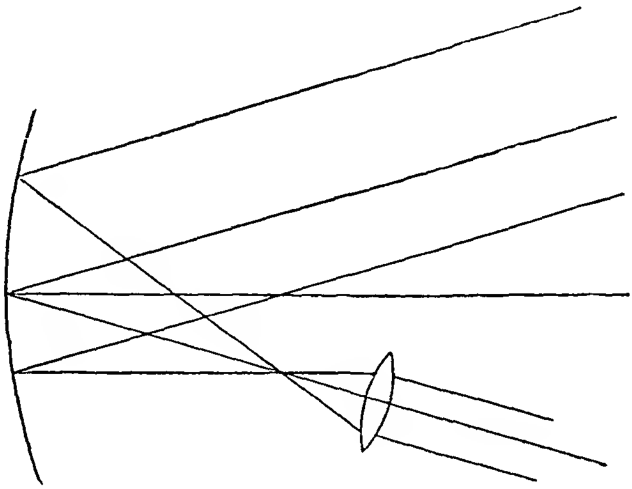


FIG. 120.

image formed by the mirror is viewed directly with an eye-piece. The incidence is made slightly oblique, and the eye-piece placed a little to the side, so that the head of the observer may be as much out of the way as possible.

In *Newton's* telescope the incidence on the concave mirror is direct, and a small plane mirror or reflecting prism is used to reflect the image out at the side, where the eye-piece receives it.

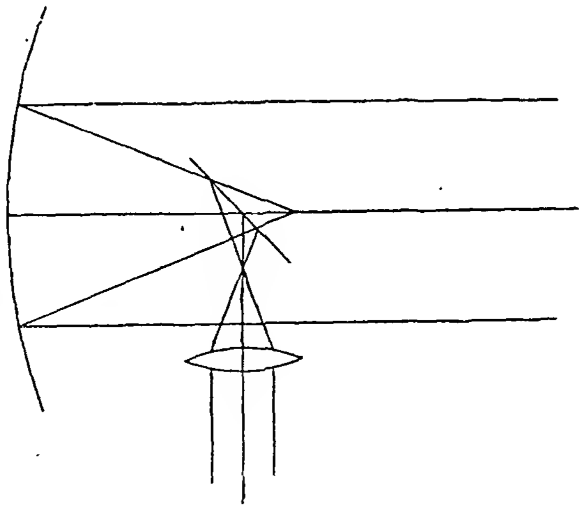


FIG. 121.

In *Gregory's* telescope (Fig. 122) the light, after reflexion at the mirror, forms an image, $a' b'$. This image forms, with the help of another concave mirror, an image, $a'' b''$. This is viewed by the eye-piece. Adjustment for different eyes is effected by moving the small mirror along the axis.

EXAMPLES. 105.

Find, by Practice, the value of

1. 7 md. 15 seers at ₨3. 7*a*. 8*p*. per md.
2. 9 md. 17½ seers at ₨4. 10*a*. 8*p*. per md.
3. 27 cwt. 2 qr. 7 lb. at £3. 7*s*. 6*d*. per cwt.
4. 11 tons 14 cwt. at £5. 17*s*. 6*d*. per ton.
5. 17 tons 15 cwt. 2 qr. 21 lb. at £3. 15*s*. 9*d*. per cwt.
6. 6 tons 3 cwt. 2 qr. 24 lb. at 17*s*. 7*d*. per cwt.
7. 2 tons 13 cwt. 3 qr. 7 lb. at £1. 1*s*. 4*d*. per cwt.
8. 3 md. 27 seers 8 ch. at ₨10. 5*a*. 8*p*. per md.
9. 7 md. 18 seers 9 ch. at ₨13. 7*a*. 5*p*. per md.
10. 8 md. 3 seers 12 ch. at 3*a*. 4*p*. per seer.
11. 1 md. 17 seers 10 ch. at 7*a*. 6*p*. per seer.
12. 4 cwt. 3 qr. 14 lb. at £1. 13*s*. 4*d*. per ton.
13. 7 cwt. 2 qr. 21 lb. at £6 per ton.
14. 3 tons 17 cwt. 3 qr. 13 lb. 12 oz. at £1. 18*s*. 9*d*. per cwt.
15. 3 md. 37 seers 12 ch. at 7*s*. 6*d*. per seer.
16. 2 tons 7 cwt. 1 qr. 13 lb. 14 oz. at ₨9. 11*a*. per qr.
17. 7 sacks of flour, each 3 md. 15 seers, at ₨7. 10*a*. per md.
18. 24 bales of cotton, each 5 cwt. 2 qr., at 16*s*. 7¾*d*. per cwt.
19. 35 chests of tea, each 1 md. 17 seers 9 ch., at ₨80. 12*a*. per md.
20. 321 boxes of coffee, each 1 cwt 2 qr. 21 lb., at £7. 18*s*. per cwt.
21. Find the total produce of a field of 3 ac. 3 ro. 25 po. at 3 qr. 6 bus. 2 pk. per acre.
22. Find the produce of 2 ac. 2 ro. 88 sq. yd. at 7 cwt. 3 qr. 14 lb. per acre.
23. Find the price of 29 yd. 2 ft. 9 in. of silk at 7*s*. 10½*d*. per yd.
24. Find the weight of 231 bales of cloth, each weighing 2 cwt. 2 qr. 14 lb.
25. Find the weight of 329 boxes, each weighing 7 md. 27½ seers.
26. Find the tax on £329. 15*s*. at 1*s*. 7½*d*. in the £.
27. Find the tax on ₨3090. 8*a*. at 1*a*. 4½*p*. in the ₨.
28. Find the cost of 5 qr. 3 bus. 2 pk. of oats at £2. 14*s*. 4*d*. per qr.

29. Find the price of 12 gall. 3 qt. $1\frac{1}{2}$ pt. of milk at ₨3. 8*a.* per gallon.

30. Find the value of 225 cwt. at £21. 5*s.* 7*d.* per ton.

31. Find the value of 257 things, 10 of which cost ₨3. 9*a.* 4*p.*

32. Find, to the nearest pie, the rent of 275·365 bighas at ₨3. 7*a.* 9*p.* per bigha.

33. Find the value of 1 ton 11 cwt. 1 qr. 11 lb. at £6·285 per ton.

34. Find the dividend on ₨5146. 12*a.* at 14*a.* 6*p.* in the ₨.

35. If a man's debts amount to ₨37925. 14*a.*, and he can pay only 3*a.* $4\frac{1}{2}$*p.* for each rupee, how much do his creditors get?

XXX. SQUARE ROOT.

167. A number is called the **square root** of its square. Thus 2 is the square root of 4; 3 is the square root of 9.

The square root of a number is indicated by the symbol $\sqrt{}$ placed before it. Thus $\sqrt{4}$ indicates the *square root of* 4, that is, 2.

168. A number whose square root can be expressed exactly either by a whole number or by a fraction is called **a perfect square.**

Note. It may be noticed that, no number, integral or decimal, which ends with 2, or 3, or 7, or 8, is a perfect square.

169. When the square root of a whole number which is a perfect square does not exceed 20, we obtain it from the multiplication table. Thus from the table we know that the square root of 81 is 9; of 169 is 13. We have, however, a rule by which we can find the square root of any number consisting of more than two figures

170. We observe that the square root of 100 is 10, of 10,000 is 100, of 1,000,000 is 1,000; and so on. Hence it follows that the square roots of numbers less than 100 consist of only *one* figure in their integral parts; of numbers between 100 and 10,000, of two figures in their integral parts; of numbers between 10,000 and 1,000,000, of three figures in their integral parts; and so on. If then a point be placed over every second figure in any number beginning with the *units'* the number of points will be the same as the number of figures in the integral part of the square root. Thus the square root of $3\dot{1}3\dot{6}$ consists of two figures in its integral part; the square root of $\dot{1}5\dot{6}2\dot{5}$ consists of three figures in its integral part.

171. Now suppose we have to extract the square root of 3136.

We first divide the number into periods of two figures each, by placing dots over every second figure beginning with the units.*

```
      3136 ( 56
      25
106 ) 636
      636
```

We then find the greatest number (5) whose square is contained in the first period ; this is the first figure of the root ; then subtract its square (25) from the first period and to the remainder (6) bring down the second period, thus getting 636 for the new dividend. Next, we divide this number omitting the last figure, by twice the part of the root already found (*i.e.*, we divide 63 by 10), and annex the quotient (6) to the root and also to the *trial divisor* (10) ; then multiply the divisor as it now stands (106) by the figure of the root last found. Now, subtracting this product from 636, we have no remainder : and we conclude that 56 is the square root of 3136.

If there be more periods to be brought down, the above operation must be repeated, as in the annexed example.

```
      15625 ( 125
      1
 22 ) 56
      44
245 ) 1225
      1225
```

Here, after two figures in the root have been obtained, the remainder is 12 ; to this we bring down the third period, thus getting 1225 as the last dividend. We divide this number, last figure omitted, by twice the part of the root already found (*i.e.*, we divide 122 by 24), getting 5 as the quotient. We then annex 5 to the root and also to the trial divisor 24 : etc.

172. In obtaining the *second* figure of the root by division we sometimes get a quotient which is too large. In such a case we find the root-figure by trial, as in the two following examples.

(i)

```
     225 ( 15
     1
25 ) 125
     125
```

Here, dividing 12 by 2, the quotient is 6. Taking 6 as the required figure we find that the product (26×6) is greater than 125. We then take 5 which is found to be the required root-figure.

(ii)

```
     361 ( 19
     1
29 ) 261
     261
```

Here, division gives 13 which is obviously inadmissible. By trial we find 9 to be the required root-figure.

* *N. B.* Each period consists of the figure over which a dot is placed and the figure to its left. Here the first period is 31 and second 36. The *first* period may consist of only one figure.

173. When the trial divisor is greater than the number to be divided by it (or when the quotient is 1 but found too large) we set down 0 in the root, annex 0 to the divisor, bring down the next period, and proceed in the usual way. The two following examples are given for illustration.

(i)

$$
\begin{array}{r|l}
 & \dot{4}12\dot{0}\dot{9}\ (\ 203 \\
 & \underline{4} \\
403\ ) & 1209 \\
 & \underline{1209}
\end{array}
$$

(ii)

$$
\begin{array}{r|l}
 & \dot{4}4\dot{0}1\dot{6}0\dot{4}\ (\ 2098 \\
 & \underline{4} \\
409\ ) & 4016 \\
 & \underline{3681} \\
4188\ ) & 33504 \\
 & \underline{33504}
\end{array}
$$

174. In the process of extracting the square root, a remainder is often left, which is greater than the divisor. In the following example the second remainder 35 is greater than the divisor 29.

$$
\begin{array}{r|l}
 & \dot{3}9\dot{6}0\dot{1}\ (\ 199 \\
 & \underline{1} \\
29\ ) & 296 \\
 & \underline{261} \\
389\ ) & 3501 \\
 & \underline{3501}
\end{array}
$$

EXAMPLES. 106.

Find the square root of

1. 441.	**2.** 576.	**3.** 729.	**4.** 961.
5. 1024.	**6.** 6561.	**7.** 5625.	**8.** 9216.
9. 27225.	**10.** 54756.	**11.** 49284.	**12.** 18225.
13. 119025.	**14.** 193600.	**15.** 646416.	**16.** 717409.
17. 4937284.	**18.** 2819041.	**19.** 1002001.	**20.** 1522756.
21. 82264900.	**22.** 62504836.	**23.** 97535376.	**24.** 21224449.
25. 3226694416.	**26.** 6407522209.	**27.** 236144689.	
28. 360117609604.	**29.** 295066240000.	**30.** 15241578750190521.	

31. A certain number of men spent R1681, each spending as many rupees as there were men; how many men were there?

32. A certain number of persons agree to subscribe as many pies each as there are subscribers; the whole subscription being R33. 5*a*. 4*p*. How many subscribers were there?

33. A gardener plants an orchard with 5776 trees and arranges them so that the number of rows of the trees equals the number of trees in each row. How many rows were there?

34. A general having 11025 men under him, arranges them into a solid square. Find the number of men in the front.

35. A general wishing to arrange his men, who were 63510 in number, into a solid square, found that there were 6 men over. How many men were there in the front?

36. Find the least integer which must be subtracted from 4230 in order to become a perfect square.

175. When a number, which is a perfect square, can be easily separated into prime factors, its square root may be found by inspection.

Thus $\sqrt{8100}=\sqrt{2^2\times 5^2\times 3^2\times 3^2}=2\times 5\times 3\times 3=90.$

Example. What is the smallest whole number by which 1260 must be multiplied in order to become a perfect square?

Since $1260=2^2\times 3^2\times 5\times 7$, $\therefore$ the number required $=5\times 7=35$.

EXAMPLES. 107.

Find, by factors, the square root of

1.	900.	2.	1600.	3.	324.	4.	576.
5.	1296.	6.	4096.	7.	1764.	8.	7056.
9.	11025.	10.	53361.	11.	99225.	12.	571536.

13. $27\times 12\times 14\times 56$.

14. $182\times 77\times 66\times 39$.

15. $609\times 290\times 165\times 154$.

16. Find the smallest whole number by which 450 must be multiplied in order to become a perfect square.

17. Find the least number by which 2940 must be multiplied in order to become a perfect square.

18. Find the least number by which 968 must be divided in order to become a perfect square.

19. Find the least square number which is divisible by 10, by 16 and by 24.

20. What must be the least number of soldiers in a regiment, that will allow it to be drawn up 10, 15 or 25 deep, and also to be formed into a solid square?

176. *To find the square root of a Decimal Fraction.*

To find the square root of a decimal fraction we proceed as in the case of a whole number. In pointing, the first point must be

placed or supposed to be placed on the units' figure. In the root the decimal point must be placed immediately after the root-figures corresponding to the integral part of the number.

We observe that if any decimal be squared there will be an even number of decimal places in the result. Consequently a decimal fraction (in its simplest form) to be a perfect square must have an even number of decimal places, and the number of decimal places in the root must be one-half of the number in the square.

If the given decimal is not a perfect square (which is always the case when the decimal in its simplest form contains an odd number of decimal places) the square root will be a non-terminating decimal; and we can find the square root to any number of decimal places we like.

In finding the square root of a decimal, the number of decimal places in it must be made *even*, by annexing ciphers, if necessary.

Example 1. Find the square roots of 11·9025 and ·5625.

```
   11·9025 ( 3·45   Ans.          ·5625 ( ·75   Ans.
    9                              49
64 ) 290                      145 ) 725
     256                            725
685 ) 3425
      3425
```

Example 2. Find the square root of ·045 to three places of decimals.

Here, we are to have three decimal places in the root; therefore in the given number, we make the decimal places *six*.

```
    ·045000 ( ·212...   Ans.
     4
41 ) 50
     41
422 ) 900
      844
       56
```

Example 3. Find the square root of 3 to two places of decimals.

```
     3·0000 ( 1·73...   Ans.
     1
27 ) 200
     189
343 ) 1100
      1029
        71
```

EXAMPLES. 108.

Find the square root of

1. 11·56. 2. 4·7089. 3. 39·0625. 4. 82·4464.
5. ·0064. 6. ·005329. 7. 1082·41. 8. 5·774409.
9. ·00053361. 10. ·00002025. 11. 236·144689. 12. ·804609.
13. ·0000034188O1. 14. 1·002001. 15. 938703·06991561.

Find to four places of decimals the square root of

16. 761·9. 17. 1·7. 18. 237·615. 19. 5. 20. 876·535.
21. ·1. 22. ·5. 23. 23·1. 24. ·9. 25. 20.
26. ·016. 27. ·00064. 28. 7. 29. 66. 30. 13.

177. *To find the square root of a Vulgar Fraction.*

The square root of a vulgar fraction is the square root of its numerator divided by the square root of its denominator.

Example 1. $\sqrt{\frac{16}{25}}=\frac{\sqrt{16}}{\sqrt{25}}=\frac{4}{5}.$

Example 2. $\sqrt{2\frac{1}{4}}=\sqrt{\frac{9}{4}}=\frac{\sqrt{9}}{\sqrt{4}}=\frac{3}{2}=1\frac{1}{2}.$

Example 3. $\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{\sqrt{4}}=\frac{1\cdot73\ldots}{2}=\cdot86\ldots$

If the denominator be not a perfect square it is advantageous to make it so by multiplication.

Example 4. $\sqrt{\frac{1}{6}}=\sqrt{\frac{1\times6}{6\times6}}=\frac{\sqrt{6}}{\sqrt{36}}=\frac{2\cdot449\ldots}{6}=\cdot408\ldots$

Example 5. $\sqrt{\frac{5}{18}}=\sqrt{\frac{10}{36}}=\frac{\sqrt{10}}{\sqrt{36}}=\frac{3\cdot1622\ldots}{6}=\cdot5270\ldots$

Note. The square root of a fraction can also be found by reducing the fraction to a decimal and then extracting the square root of the decimal.

EXAMPLES. 109.

Find the square root of

1. $\frac{196}{5625}$. 2. $5513\frac{1}{16}$. 3. $32\frac{32}{49}$. 4. $101\frac{1}{400}$. 5. $\frac{8}{4\frac{1}{2}}$.
6. $2\cdot\dot{7}$. 7. $28\cdot\dot{4}$. 8. $3\cdot36\dot{1}$. 9. $8\cdot02\dot{7}$. 10. $\cdot07\dot{1}$

Find to 3 places of decimals the square root of

11. $\frac{7}{4}$. 12. $\frac{5}{7}$. 13. $\frac{2}{3}$. 14. $\frac{5}{8}$. 15. $\frac{7}{12}$.

16. $\cdot\dot{3}$. **17.** $\cdot 41\dot{6}$. **18.** $\dfrac{1\cdot 23}{\cdot 5}$. **19.** $\dfrac{1}{2\cdot 5}$. **20.** $\dfrac{5\cdot 04}{\cdot 012}$.

21. Simplify $\sqrt{(75\frac{1}{9})} \times \sqrt{(1\cdot\dot{7})} \div \sqrt{(2\frac{34}{81})}$.

178. When *more than half* the number of figures of a square root has been obtained by the ordinary method, the remaining figures may be obtained by division only.

Example 1. To find the square root of 189475225.

Here we find the first three figures in the ordinary way. To find the remaining two figures by division, we take twice the part of the root already found, as the divisor; we bring down one figure to the last remainder and divide; then to the new remainder bring down the next figure and divide. The quotient thus obtained gives the two remaining figures of the root.

```
        18947|5225 ( 137|65  Ans.
        1
  23 )  89
        69
 267 )  2047
        1869
 274 )   1785 ( 65
         1644
          1412
          1370
            42
```

Note. Of course this process does not show whether the given number is a perfect square or not, but it is very useful in cases like the following.

Example 2. Find the square root of 2 to seven places of decimals.

Here we find 5 figures of the root by the ordinary method and the remaining three by division.

```
           2·  ( 1·41421|135...  Ans.
           1
    24 )  100
           96
   281 )   400
           281
  2824 )   11900
           11296
 28282 )    60400
            56564
 28284 )     38360 ( 135
             28284
             100760
              84852
              159080
              141420
               17660
```

EXAMPLES. 110.

Find to 6 places of decimals the square root of

1. 5.	2. 17.	3. 761·9.	4. ·0003841.
5. $\frac{3}{5}$.	6. 3.	7. ·07.	8. ·85.
9. 7619.	10. $\frac{3}{8}$.	11. 237·615.	12. 1·7.
13. $\frac{5}{7}$.	14. 23·8369.	15. ·000943.	16. 10.

XXXI. CUBE ROOT.

179. A number is called the **cube root** of its **cube**. Thus 2 is the cube root of 8 ; 3 is the cube root of 27.

The cube root of a number is indicated by the symbol $\sqrt[3]{\ }$ placed before it. Thus $\sqrt[3]{8}$ indicates the *cube root of* 8, *i.e.*, 2.

A number whose cube root can be expressed exactly either by a whole number or by a fraction is called a **perfect cube**.

The cubes of	1,	2,	3,	4,	5,	6,	7,	8,	9,
are respectively	1,	8,	27,	64,	125,	216,	343,	512,	729.

[These results should be committed to memory].

180. The method of finding the cube root of a number is as follows :

Example 1. Find the cube root of 13824.

$$\begin{array}{r|l} & 1\dot{3}82\dot{4}\ (\ 24\ \textit{Ans.} \\ & 8 \\ \hline 2^2\times 300 = 1200 & 5824 \\ 2\times 30\times 4 = 240 & \\ 4^2 = 16 & \\ \hline 1456 & 5824 \\ \hline \end{array}$$

We divide the number into periods of 3 figures each, the number of dots indicating the number of figures in the cube root.

We find that 2 is the highest number whose cube is less than the first period ; this then is the first figure of the root. We subtract the cube of 2 from the first period and to the remainder we bring down the second period.

Next, we multiply the square of 2 (the first figure of the cube root) by 300 and set down the product 1200 ; this is the *trial divisor.* Dividing 5824 by this, the quotient is 4 ; this is the second figure of the root. Now we multiply the first figure of the cube root by 30 and this product by the second figure of the root, and set down

the result under the trial divisor; then set down under this the square of the second figure of the cube root. Adding these three we get 1456 as our divisor. We then multiply this by the second figure of the root and subtracting the product from 5824 we find that there is no remainder. Therefore we conclude that 24 is the cube root of 13824.

If the cube root contains three or more figures the above process must be repeated.

Example 2. Find the cube root of 33076161.

Process: 33076161 (321 *Ans.*

			27
$3^2 \times 300$	=	2700	6076
$3 \times 30 \times 2$	=	180	
2^2	=	4	
		2884	5768
$32^2 \times 300$	=	307200	308161
$32 \times 30 \times 1$	=	960	
1^2	=	1	
		308161	308161

Note. Remarks of Arts. 172, 173 and 174 with regard to the process of extraction of the square root apply equally to the process of extraction of the cube root.

EXAMPLES. III.

Find the cube root of

1. 1331. **2.** 15625. **3.** 46656. **4.** 110592.
5. 117649. **6.** 373248. **7.** 2197. **8.** 185193.
9. 704969. **10.** 912673. **11.** 15069223. **12.** 105823817.
13. 843908625. **14.** 873722816. **15.** 219365327791.
16. 167284151. **17.** 7311189187729. **18.** 10970645048.
19. 93162981941037. **20.** 1371742108367626890260631.

181. A decimal fraction (in its simplest form) to be a perfect cube must have 3, 6, 9,... decimal places; that is, the number of decimal places in it must be some multiple of 3. If the number of decimal places be not a multiple of 3, the cube root can be obtained to any number of decimal places we like. In extracting the cube root of a decimal, the number of decimal places must be made a multiple of 3 by annexing ciphers, if necessary.

The cube root of a vulgar fraction is the cube root of its numerator divided by the cube root of its denominator.

EXAMPLES. 112.

Find the cube root of

1. 17 576.	2. 132·651.	3. ·493039.
4. 64481·201.	5. 18·609625.	6. ·007645373.
7. ·876467493.	8. ·001030301.	9. $\frac{64}{343}$.
10. $\frac{729}{140608}$.	11. $49\frac{8}{27}$.	12. $7558\frac{197}{512}$.
13. $\cdot\dot{0}3\dot{7}$.	14. $1587\cdot\dot{9}6\dot{2}$.	15. $3845\cdot\dot{2}9\dot{6}$.
16. $46\frac{82}{125}$.	17. $20\frac{51}{64}$.	18. $2\cdot\dot{3}7\dot{0}$.

Find to three places of decimals the cube root of

19. 3·539.	20. 11.	21. 24.	22. 7·52	23. ·8.
24. ·27.	25. $\frac{1}{10}$.	26. $\frac{1}{6}$.	27. ·0047.	28. $5\frac{4}{7}$.

182. When at least *one more than half* the number of figures in the cube root of a number has been found by the ordinary method, the remaining figures of the root may be found by division only.

Note. In this case we take for the divisor 300 times the square of the part of the cube root already obtained and proceed exactly as in Art. 178.

EXAMPLES. 113.

Obtain to 6 places of decimals the cube root of

1. 3·539.	2. 24.	3. 7·52.
4. ·002.	5. ·003.	6. $18\frac{7}{12}$.

183. The **fourth root** of a number is found by taking the square root of the square root of the number.

The **sixth root** of a number is found by taking the cube root of the square root of the number.

The **ninth root** of a number is found by taking the cube root of the cube root of the number.

EXAMPLES. 114.

Find the fourth root of

1. 256.	2. 234256.	3. 1679616.	4. 1575·2961.

Find the sixth root of

5. 531441. 6. 308915776. 7. 24794911296.

Find the ninth root of

8. 262144. 9. 1953125. 10. 3000.

XXXII. MEASUREMENT OF AREA.

184. In Arithmetic we consider the areas of **rectangles** only.

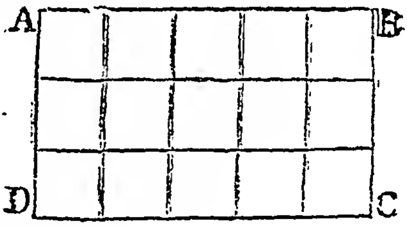
Example. The floor, the ceiling and each wall of an ordinary room ; a sheet of paper ; each side of an ordinary box or brick ; all these are rectangular surfaces.

The length and breadth of a rectangle are called its **dimensions**.

185. The **unit of area** is a square whose side is the unit of length.

Area or **Surface** is measured by the number of units of area which it contains ; just as a length is measured by the number of units of length which it contains.

186. *To find the area of a rectangle.*



Let $ABCD$ be a rectangle, of which the length AB is 1 yd. 2 ft., and the breadth AD is 3 ft. Then, if the unit of length be a foot, the measure of AB is 5 and of AD is 3.

Divide AB and AD into 5 and 3 equal parts respectively, and through the points of division draw lines parallel to AD, AB respectively. Then the rectangle $ABCD$ is divided into 5×3 equal squares, the side of each of which is a foot in length.

Now, each of these squares is the unit of area ; therefore the *measure* of the area $ABCD$ (which is the same as the number of these squares) is 5×3 or 15.

$\therefore$ Area of $ABCD = 15$ sq. ft.

And generally, in any rectangle,

measure of area = measure of length $\times$ measure of breadth ;

or, more briefly,

area = length $\times$ breadth.

Whence,

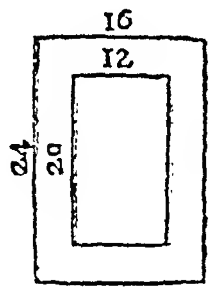
length = area $\div$ breadth ;
breadth = area $\div$ length.

Note. A **square foot** is a square whose side is a foot. Note the difference between "3 **square feet**" and "3 **feet square**." *Three square feet* denotes an area 3 times as large as a square foot ; *three feet square* denotes the area of a square whose side is 3 feet.

Example 1. Find the area of the floor of a room 10 ft. 6 in. long and 6 ft. 4 in. broad.

$$\begin{aligned}\text{Length of room} &= 10\tfrac{1}{2}\text{ ft.};\\ \text{breadth }\ ,,\ \ ,, &= 6\tfrac{1}{3}\text{ ft.};\\ \therefore\ \text{area }\ ,,\ \ ,, &= 10\tfrac{1}{2}\times 6\tfrac{1}{3}\text{ sq. ft.}\\ &= \tfrac{21}{2}\times\tfrac{19}{3}\text{ sq. ft.}\\ &= \tfrac{133}{2}\text{ sq. ft.}\\ &= 66\text{ sq. ft. }72\text{ sq. in.}\end{aligned}$$

Example 2. A rectangular court, 24 yards long and 16 yards broad, has within it a path of uniform breadth of 2 yards running round it ; find the area of the path.



$$\begin{aligned}\text{Area of court} &= 24\times 16\text{ sq. yd.}\\ &= 384\text{ sq. yd.}\end{aligned}$$

The path takes off $(2+2)$ yd. from the length and $(2+2)$ yd. from the breadth ;

$\therefore$ length of inner court $= 20$ yd.,

and breadth ... ... $= 12$ yd. ;

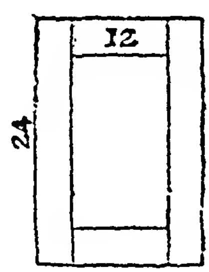
$\therefore$ area ... ... $= 20\times 12$ sq. yd.

$= 240$ sq. yd.

$\therefore$ area of path $=(384-240)$ sq. yd.

$=144$ sq. yd.

Or thus :



Length of the path

$=(24\times 2+12\times 2)$ yd.

$=72$ yd. ;

$\therefore$ the area of the path $=72\times 2$ sq. yd.

$=144$ sq. yd.

Example 3. Find the breadth of a courtyard 41 sq. ft. 80 sq. in. in area, and 7 ft. 4 in. in length.

$$\text{Area} = (41 + \tfrac{80}{144}) \text{ sq. ft.}$$
$$= 41\tfrac{5}{9} \text{ sq. ft.};$$
$$\text{length} = 7\tfrac{1}{3} \text{ ft.}$$
$$\therefore \quad \text{breadth} = \frac{41\frac{5}{9}}{7\frac{1}{3}} \text{ ft.} = \frac{374}{9} \times \frac{3}{22} \text{ ft.} = 5\frac{2}{3} \text{ ft.}$$
$$= 5 \text{ ft. } 8 \text{ in.}$$

Example 4. How many paving stones, each 2 ft. 8 in. long and 17 in. wide, will cover the courtyard in *Ex.* 3?

$$\text{Area of court} = 41\tfrac{5}{9} \text{ sq. ft.};$$
$$\text{area of a stone} = 2\tfrac{2}{3} \times \tfrac{17}{12} \text{ sq. ft.} = \tfrac{34}{9} \text{ sq. ft.};$$
$$\therefore \quad \text{number of stones rqd.} = \frac{41\frac{5}{9}}{\frac{34}{9}} = \frac{374}{9} \times \frac{9}{34} = 11.$$

Example 5. Find the cost of matting the room in *Ex.* 1, at 3 annas per sq. ft.

The cost may be found by Practice or by Compound Multiplication.

EXAMPLES. 115.

Find the area of the rectangles having the following dimensions:

1. 15 ft. by 12 ft.
2. 20 ft. by 16 ft.
3. 13 ft. 6 in. by 8 ft. 8 in.
4. 9 ft. 10 in. by 6 ft. 7 in.
5. 10 ft. $7\frac{1}{2}$ in. by 7 ft. $4\frac{1}{2}$ in.
6. 9 yd. 2 ft. by 7 yd. 1 ft.

Find the breadth of a room whose

7. area = 363 sq. ft., and length = 33 ft.

8. area = 6 sq. ft. 60 sq. in., and length = 2 ft. 9 in.

9. area = 5 ac. 1 ro. 36 po., and length = 267 yd. 2 ft.

10. area = 94 sq. yd. 8 ft. 84 in., and length = 32 yd. 1 ft. 8 in.

11. Find the area of a square field whose side is 32 ft. 8 in.

12. Find the area of a square room whose side is 3 yd. 2 ft. 3 in

13. How many paving stones, each $1\frac{1}{2}$ ft. by 9 in., would be required to pave a square courtyard whose side is 21 ft.?

14. How many pieces of carpet, each 5 ft. long and 3 ft. wide, will cover the floor of a room 20 ft. by 13 ft. 6 in.?

15. Find the cost of carpeting a room, 10 ft. 6 in. by 6 ft. 6 in., at Rs.2 per sq. ft

16. Find the cost of polishing a marble slab, 3 ft. 3 in., by 2 ft. 6 in., at 2*d.* per sq. in.

17. A room, 20 ft. long, 16 ft. broad, has a stained border all round it 2 ft. wide; what is the area of the stained part?

18. A rectangular piece of ground is 88 yards long and contains an acre; it consists of a walk 6 ft. wide surrounding a grass-plot: find the area of the walk.

19. How many stone slabs, 3 ft. long, 1 ft. wide, are requisite for paving a path which encloses a rectangular garden half a mile long and quarter of a mile wide, the path being 6 ft. wide?

20. A gravel path 5 ft. wide runs round a rectangular garden, 100 yd. by 75 yd.; find the cost of making it at 4*a.* 6*p.* per sq. yd.

21. How many sq. yards of matting will be wanted to cover a room 31 ft. 6 in. by 22 ft. 6 in.? What will be the cost at 4*d.* per sq. yd.?

22. If 1200 stones, each 2 feet square, will pave a court, find the area of the court.

23. The cost of varnishing the floor of a room, 24 ft. long, at 2*s.* 6*d.* per sq. yd., is £5; find the breadth of the room.

24. A garden roller is 3 ft. 3 in. wide, and its circumference is 6 ft. 9 in.; how many sq. ft. of ground does it pass over in one complete revolution?

25. A sheet of paper is 20 in. long and 18 in. wide; by how much must the width be narrowed to leave a surface of $2\frac{1}{4}$ sq. ft.?

26. What length must be cut off a plank which is $5\frac{1}{4}$ in. broad, that the area may be a sq. foot?

27. A factory has 100 windows, 60 of which severally contain 8 panes, each 9 in. by 6 in.; and the remainder severally contain 10 panes, each 2 ft. square; find the cost of glazing the whole at 10 annas per sq. ft.

28. What must be the length of a piece of land, 15 yards wide, that can be exchanged for a piece of the same quality, measuring 20 yards each way?

29. Find the area of the square which has the same perimeter as a rectangle whose length is 48 ft. and is 3 times its breadth.

30. How many flag-stones, each 5·76 ft. long and 4·15 ft. wide, are requisite for paving a cloister, which encloses a rectangular court, 45·77 yd. long and 41·93 yd. wide, the cloister being 12·45 ft. wide?

31. A room measuring 42 ft. 6 in. by 22 ft. 9 in. inside, with walls 2 ft. 3 in. thick, is surrounded by a verandah 10 ft. 6 in. wide. Find the cost of paving this verandah with tiles measuring $4\frac{1}{2}$ in. by 3 in., and costing 6 pies each.

187. *Example* 1. Find the length of the side of a square which contains 91 sq. ft. 121 sq. in.

Area $=$ 91 sq. ft. 121 sq. in. $=$ 13225 sq. in.;

$\therefore$ length of side $= \sqrt{13225}$ in. $=$ 115 in.
$= 9$ ft. 7 in.

Example 2. Find the diagonal of a rectangular field, 16 yd. long and 12 yd. wide.

By Euclid I. 47,

the diagonal $= \sqrt{16^2+12^2}$ yd. $= \sqrt{256+144}$ yd.
$= \sqrt{400}$ yd. $=$ 20 yd.

Example 3. The area of a room which is twice as long as it is broad is 26 sq. yd. 8 sq. ft.; how long is it?

The room can be divided into two equal squares whose side is equal to the breadth of the room.

Area of each square $=$ 13 sq. yd. 4 sq. ft.
$=$ 121 sq. ft.;

$\therefore$ side of each square $= \sqrt{121}$ ft. $=$ 11 ft.;

$\therefore$ breadth of room $=$ 11 ft. $=$ 3 yd. 2 ft.;

$\therefore$ length of room $=$ 7 yd. 1 ft.

EXAMPLES. 116.

1. The area of a square field is 10 acres; find the length of its side.

2. The area of a square room is 502 sq. ft. 73 sq. in.; find the length of each side.

3. How many yards of fencing are required to enclose a square garden containing 4 ro. 1 po. 29 yd. $6\frac{3}{4}$ ft.?

4. A rectangular field is 40 yards long and 30 yards broad; find the distance from corner to corner.

5. What is the length of the diagonal of a square whose side is 4 yards?

6. The area of a square is 900 sq. ft.; what is the length of its diagonal?

7. The area of the floor of a room is 162 sq. ft.; its length is twice its breadth; find its length.

8. Find the length of a rectangular field which is 3 times as long as it is broad and which contains 768 sq. yd.

9. A room is half as long again as it is broad and its area is 69·36 sq. yd.; find its perimeter.

10. The sides of two squares contain 77 yd. 1 ft. 9 in. and 7 yd. 2 ft. 4 in. respectively; find the side of a square whose area is equal to the sum of the areas of the two squares.

188. Carpeting the floor and papering the walls of a room.

Example 1. Find the length of carpet $2\frac{1}{3}$ ft. wide, required for a room 28 ft. long, 20 ft. broad.

The carpet which will cover the floor of a room has the same area as the floor.

$$\text{Area of floor} = 28 \times 20 \text{ sq. ft.};$$

$$\therefore \text{ length of carpet reqd.} = \frac{28 \times 20}{2\frac{1}{3}} \text{ ft.} = \frac{28 \times 20 \times 3}{7} \text{ ft.}$$

$$= 240 \text{ ft.} = 80 \text{ yd.}$$

Example 2. Find the area of the four walls of a rectangular room 20 ft. long, 15 ft. broad and 10 ft. high.

The area of the four walls of a rectangular room is obtained by multiplying the circuit (*i.e.*, twice the sum of length and breadth) of the room by the height of the room.

$$\text{The circuit} = (20 + 15) \times 2 \text{ ft.} = 70 \text{ ft.};$$

$$\therefore \text{ the area of walls} = 70 \times 10 \text{ sq. ft.} = 700 \text{ sq. ft.}$$

To find the length of paper required to cover the walls, proceed as in the preceding example.

Note. In estimating the length of paper required, deductions for doors, windows and fireplaces must be made.

N. B. The cost of carpet or paper may be found by Practice or by Compound Multiplication.

EXAMPLES. 117.

Find the length of carpet required for rooms having the following dimensions:

1. Room, 25 ft. long, 18 ft. broad; carpet, 2 ft. 6 in. wide.
2. Room, 20 ft. long, 12 ft. 6 in. broad; carpet, 27 in. wide.
3. Room, $30\frac{3}{4}$ ft. long, $20\frac{1}{2}$ ft. broad; carpet, 42 in. wide.

Find the expense of carpeting a room,

4. 16 ft. by 10 ft., with carpet 3 ft. wide, at R2. 8*a*. a yard.

5. 30 ft. 9 in. by 25 ft., with carpet 30 in. wide, at 4*s*. 6*d*. a yard.

Find the area of the walls of the following rectangular rooms:

6. Length 20 ft., breadth 16 ft., height 9 ft.

7. Length 15 ft. 6 in., breadth 12 ft., height 9 ft.

8. Length 21 ft. 7 in., breadth 16 ft. 5 in., height $3\frac{1}{2}$ yd.

Find the length of wall paper required for the following rooms:

9. 25 ft. long, 20 ft. wide, 12 ft. high; paper 15 in. wide.

10. 14 ft. long, 10 ft. wide, 7 ft. high; paper 14 in. wide.

11. 27 ft. long, 18 ft. wide, 10 ft. high; with paper 16 in. wide, allowing for 2 doors each 7 ft. by 4 ft.

12. 28 ft. long, 20 ft. broad, $9\frac{1}{2}$ ft. high; with paper 20 in. wide, allowing for a door 6 ft. by $3\frac{1}{2}$ ft. and a window 3 ft. by $2\frac{1}{2}$ ft.

Find the expense of papering rooms whose dimensions are:

13. Length 21 ft., breadth 16 ft., height 10 ft.; with paper 16 in. wide, at 4*a*. a yard.

14. Length 50 ft., breadth 35 ft., height 15 ft.; with paper 15 in. wide, at 6*d*. a yard.

15. Length 18 ft., breadth 16 ft., height 9 ft.; with paper 15 in. wide, at 9*d*. a yard, allowing for 3 doors each 6 ft. by $3\frac{1}{2}$ ft., 2 windows 4 ft. by $2\frac{1}{2}$ ft., and a fireplace 6 ft. by 4 ft. 6 in.

16. How many yards will remain out of 300 yards of matting 2 ft. 6 in. wide, after covering two floors, each 25 ft. 6 in. by 21 ft.?

17. A square room whose floor measures 56 sq. yd. 2 sq. ft. 36 sq. in., is 10 ft. 4 in. high; find the expense of whitewashing its ceiling and walls at 2*p*. per sq. yd.

18. The cost of covering the floor of a room, $12\frac{1}{2}$ yd. by $8\frac{3}{4}$ yd., with carpet $2\frac{1}{2}$ ft. wide, is £30 . 14 . $7\frac{1}{2}$; find the price of carpet per yard.

19. It costs £2. 5*s*. to paper a room 10 yd. long and 8 yd. wide, with paper $1\frac{1}{2}$ ft. wide, at 3*d*. per yard; find the height of the room.

20. The cost of carpeting a room $16\frac{1}{2}$ ft. long and $12\frac{1}{4}$ ft. broad, with carpet at 6*s*. per yard, is £14. 17*s*.; find the width of the carpet.

21. If a postage stamp be $\frac{5}{6}$ of an inch long and $\frac{3}{4}$ of an inch broad, what will be the cost of covering the walls of a room which is 15 ft. long, 12 ft. wide and 9 ft. high, with postage stamps, 6 pies each?

22. What will be the cost of papering a room, 24 ft. long by 20 ft. broad and 8 ft. high, which has 2 doors each 7 ft. by 4 ft.,

with paper 2 ft. wide, at ₹4 a piece; the cost of putting it on being 4*a.* per piece, and each piece being 4 yards long?

23. The matting of a room, 3 times as long as broad, at 4 annas per sq. ft. cost ₹75; and the painting of the walls at 2 annas per sq. yd. cost ₹6. 6*a.* $2\frac{2}{3}$*p.*; what is the height of the room?

24. Find the expense of lining a cistern 10 ft. long, 8 ft. broad and 3 ft. deep, with lead at ₹10 per cwt., which weighs 5 lb. per sq. ft.

25. Find the cost of papering a room, 18 ft. long, 12 ft. broad and 10 ft. high, with paper 32 in. wide, at 6 annas a yard, allowing for a door 7 ft. by 4 ft., 3 windows each 4 ft. by 3 ft. and a panelling 2 ft. high round the floor.

26. A box with a lid is to be made of plank, one inch thick; the external dimensions are to be 18 in., 12 in., and 9 in.: how many sq. ft. of plank will be required?

27. The length of a room is $32\frac{1}{2}$ ft. The cost of papering the walls at ₹1. 14*a.* per sq. yd. is ₹308. 2*a.*; and the cost of carpeting the floor at ₹2. 4*a.* per sq. yd. is ₹150. 5*a.* Find the height and width of the room.

28. Find the cost of whitewashing the ceiling and the inner and outer sides of the walls of a room, 20 ft. long, 12 ft. wide and 15 ft. high, at 1 pie per sq. ft.; the walls being $1\frac{1}{2}$ ft. thick and 3 ft. higher at the outside.

LAND MEASUREMENT OF BENGAL.

189. If we have to find the area of a rectangular piece of land say, 14 bi. 3 cot. by 9 bi. 2 cot., we might proceed thus:

Area $= 14\frac{3}{20} \times 9\frac{1}{10}$ bi. (superficial) $= 128\frac{153}{200}$ bi. $=$ 128 bi. 15 cot. 4 ch. 16 ga.

But such examples are usually worked by the following rule:

Bigha multiplied by *bigha* gives *bigha.*
Bigha " " *cottah* " *cottah.*
Cottah " " *cottah* " *dhool.*

Twenty dhools make a *cottah.*

The truth of the rule will appear from the following considerations:

1 bi. × 1 bi $=$ 1 bi. (superficial);
1 bi. × 1 cot. $= 1 \times \frac{1}{20}$ bi. $= \frac{1}{20}$ bi. $=$ 1 cot. (superficial);
1 cot. × 1 cot. $= \frac{1}{20} \times \frac{1}{20}$ bi. $= \frac{1}{20}$ cot. $=$ 1 dhool.

By this method the above example will be worked thus:

We multiply all the terms of the 1st line (beginning with the lowest) by all the terms of the 2nd line (beginning with the highest).

bi.	cot.		
14 .	3		
9 .	2		
127 .	7		$=(14$ bi. 3 cot.$)\times 9$ bi.
1 .	8 .	6	$=(14$ bi. 3 cot.$)\times 2$ cot.
128 .	15 .	6	$=(14$ bi. 3 cot.$)\times(9$ bi. 2 cot.$)$

$\therefore$ Area $=128$ bi. 15 cot. 6 dhools

$=128$ bi. $15\frac{3}{10}$ cot.

$=128$ bi. 15 cot. 4 ch. 16 ga.

EXAMPLES. 118.

Find the area of the following rectangular fields:

1. 4 bi. by 3 bi.
2. 10 bi. 10 cot. by 5 bi.
3. 12 bi. 15 cot. by 8 bi. 10 cot.
4. 14 bi. 8 cot. by 14 bi. 8 cot.
5. 24 bi. 8 cot. by 14 bi. 13 cot.
6. 57 bi. 5 cot. by 42 bi. 8 cot.
7. 99 bi. 19 cot. by 49 bi. 19 cot.
8. 115 bi. 14 cot. by 105 bi. 7 cot.
9. $8\frac{1}{2}$ bi. by $3\frac{1}{4}$ bi.
10. $10\frac{3}{4}$ bi. by 15 cot.
11. 252 cubits by 164 cubits.
12. 408 cubits by 308 cubits.

XXXIII. MEASUREMENT OF SOLIDITY.

190. In Arithmetic we consider the volumes of **rectangular solids** only.

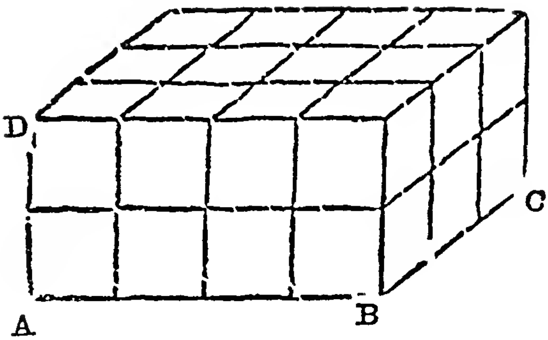
Example. A rectangular box, a brick, are rectangular solids.

The length, breadth and thickness (or height or depth) of a rectangular solid are called its **dimensions**.

191. The **unit of volume** is a cube each of whose edges is the unit of length.

Volume or **cubic content** is measured by the number of units of volume which it contains.

192. *To find the volume of a rectangular solid or rectangular parallelopiped.*



Let the annexed figure represent a rectangular parallelopiped, of which the length AB is 4 ft., breadth BC is 3 ft. and thickness AD is 2 ft. Divide AB BC, AD respectively into 4, 3 and 2 equal parts, and through the points of division draw planes parallel to the sides. Then the solid will be divided into a number of equal blocks, each of which is a *cubic foot*; and since there are two layers, in each of which there are 4×3 blocks, we see that there are $4\times 3\times 2$ blocks altogether, and the solid therefore contains $4\times 3\times 2$ cubic feet.

$\therefore$ The volume of the solid $=4\times 3\times 2$ cu. ft.

And generally, in any rectangular solid,

The measure of volume = measure of length × measure of breadth × measure of thickness.

Or, more briefly,

Volume = length × breadth × thickness.

Whence, thickness = volume ÷ (length × breadth): etc.

Example 1. Find the cubic content of a rectangular block of marble whose dimensions are 3 ft. 2 in., 2 ft. 3 in. and 1 ft. 6 in.

Volume $=3\frac{1}{6}\times 2\frac{1}{4}\times 1\frac{1}{2}$ cu. ft. $=10\frac{11}{16}$ cu. ft.

Example 2. How many bricks will be required to build a wall 20 ft. long, 10 ft. high and 2 ft. thick; each brick with its share of the mortar being 6 in. long, 3 in. wide and 2 in. deep?

$$\text{Number of bricks}=\frac{\text{volume of the wall}}{\text{volume of each brick}}=\frac{20\times 10\times 2}{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{6}}=19200.$$

Example 3. A rectangular cistern is 6 ft. long and 4 ft. broad; what is the depth of water in it, when it contains 72 cubic feet of water?

$$\text{Depth}=\frac{\text{volume of water}}{\text{area of the base}}=\frac{72}{6\times 4}\text{ ft.}=3\text{ ft.}$$

Example 4. A box with a lid is to be made of half-an-inch plank; its internal dimensions are to be 20 in., 15 in. and 9 in. How many cu. in. of wood will be required?

The external dimensions of the box are **21 in., 16 in.** and 10 in.; ∴ its external volume $=21 \times 16 \times 10$ cu. in. $=3360$ cu. in.; and its internal volume $=20 \times 15 \times 9$ cu. in. $=2700$ cu. in. ∴ Volume of wood required for the box $=(3360-2700)$ cu. in. $=660$ cu. in.

We may obtain the area of the plank required by dividing the volume of the wood by the thickness of the plank.

EXAMPLES. 119.

Find the cubic contents of the rectangular solids having the following dimensions:

1. 10 ft., 8 ft., 5 ft.
2. $7\frac{1}{2}$ ft., $5\frac{1}{4}$ ft., $4\frac{2}{3}$ ft.
3. 3 yd., 7 ft., 30 in.
4. 5 ft. 10 in., 3 ft., 6 in.
5. 7 yd. 2 ft. 9 in., 6 yd. 1 ft. 3 in., 10 ft. 10 in.
6. Find the cubic content of a cube whose edge is $3\frac{1}{2}$ ft.
7. How many pounds of water will fill a cistern 2 yd. long, 3 ft. broad and 9 in. deep, having given that a cu. ft. of water weighs 1000 oz.?
8. How many bricks, each 9 in. by 6 in. by 4 in., are required for a wall 22 yd. long, 8 ft. high and 2 ft. 6 in. thick, leaving in it a doorway 6 ft. by 4 ft.
9. How many times can a bucket, holding 2 cu. ft. of water, be filled from a tank 30 ft. long, 25 ft. wide and 10 ft. deep?
10. In what time will a cistern 16 ft. by 12 ft. by 10 ft., be filled by a pipe which discharges 40 cu. ft. of water per minute?
11. How many sheets, each 4 ft. long, 2 ft. broad and $\frac{1}{4}$ of an inch thick, can be made from 4 cu. ft. of iron?
12. Find the total weight of 27 sheets of copper, each 6 ft. long, 4 ft. broad and $\frac{1}{3}$ an inch thick, a cubic foot of copper weighing 2 cwt.
13. How many times can a pint-bottle be filled from a cistern 138·637 in. by 70 in. by 10 in., having given that a gallon contains 277·274 cubic inches?
14. A cu. inch of gold is hammered into a plate 6 in. square; find the thickness of the plate as the decimal of an inch.
15. Water is flowing into a reservoir which is 5 ft. square; how many cu. ft. of water will have flown in when the depth of water is $2\frac{1}{2}$ ft.?
16. A cistern, 12 ft. long and 8 ft. 6 in. broad, contains water; how many cu. ft. of water must be drawn off to make the surface sink half an inch?

17. A room, 40 ft. $10\frac{1}{2}$ in. by 25 ft. 8 in., accommodates 100 persons; what must be the height of the room if each person has $175\frac{2331}{3200}$ cu. ft. of air?

18. What length must be cut off a rectangular marble slab, $1\frac{1}{2}$ ft. broad and 8 in. thick, in order that it may contain 2 cu. ft.?

19. Find the cost of digging a canal 1 mile long, 6 ft. wide and 5 ft. deep, at 4 annas per cu. yd.

20. A lake, whose area is 30 acres, is covered with ice 6 inches thick; find the weight of the ice in tons, if a cubic foot of ice weigh 900 oz. avoir.

21. There are 1530 cu. ft. of air in a room 9 ft. high; find the cost of carpeting it at ₨1 per sq. ft.

22. A square room, 10 ft. high, contains 4000 cu. ft. of air: how many yards of paper, 2 ft. wide, will be required for covering its walls?

23. A solid stack, 41 ft. 8 in. by 16 ft. 8 in. by 14 ft. 7 in., contains 125000 bricks, each 10 in. long and $3\frac{1}{2}$ in. thick; find the width of each brick.

24. A piece of ground is 100 yd. long and 75 yd. wide. To what uniform depth must it be excavated that the earth taken out may form an embankment of 25000 cubic yards, supposing the earth to be increased one-ninth in volume by removal?

25. A box (with cover) is made of an-inch-and-a-half plank; its external dimensions are 4 ft., 3 ft. 6 in. and 2 ft. 3 in.: find the weight of the box, supposing a cu. ft. of the wood to weigh 36 lb.

26. The roof of a verandah is supported by 16 teak beams, each 9 ft. long, 3 in. broad and 5 in. deep. If the weight of a cubic inch of teak is $\frac{18}{25}$ of that of a cubic inch of water, and if a cubic foot of water weighs 1000 oz., find the weight in lbs. of the timber in the verandah.

27. A crow wishing to quench its thirst came to a vessel which contained 28 cu. in. of water. The crow being unable to reach the water, picked up several small stones, each three quarters of a cubic inch in size, and let them drop into the vessel until the water came to the top of the vessel. If the size of the vessel was such that it would exactly hold 73 cubic inches of water, find the number of stones dropped in by the crow.

28. The top of a tank is a rectangle whose sides are 15 ft. and 9 ft.; it is of the same horizontal section throughout its depth. What must be its depth in order that it may contain 12960 gallons of water, one gallon containing 277·274 cubic inches?

29. A moat is to be dug all round a rectangular fort, 200 yd. long and 150 yd. broad; it is to have vertical sides and to be 27 ft.

wide and 10 ft. deep throughout. Find the cost of digging it at 4 annas per cubic yard.

30. A room, 21 ft. long by $13\frac{1}{2}$ ft. wide, is surrounded by walls $1\frac{1}{2}$ ft. thick and 14 ft. high. There are two doors each $4\frac{1}{2}$ ft. by 6 ft., and one window 3 ft. by $4\frac{1}{2}$ ft. Find (i) the cost of building the walls at the rate of ₨5. 1*a.* per cubic yard, and (ii) the number of bricks, each measuring 9 in. by 4 in. by $2\frac{1}{4}$ in., required for the work.

XXXIV. DUODECIMALS.

193. Duodecimals or **Cross Multiplication** is a method (similar to that of Art. 189) of finding areas and volumes, made use of by painters, bricklayers, etc., in measuring work.

In duodecimals, the successive linear units are named and counted as follows:

1 foot = 12 *primes*; 1 prime = 12 *seconds*; 1 second = 12 *thirds*; etc.

Note. A prime = an inch. A second is often called a *part.*

The successive superficial and solid units are named and counted exactly in the same way as the linear units: Thus,

1 superficial foot = 12 superficial primes; 1 supl. prime = 12 supl. seconds; etc.

1 solid foot = 12 solid primes; 1 solid prime = 12 solid seconds; etc.

Primes, seconds, thirds, etc., are indicated by the accent ($'$), ($''$), ($'''$), etc., respectively.

The whole of the above statement may be briefly put thus:

$$\left.\begin{array}{l}1 \text{ linear foot}\\ 1 \text{ square foot}\\ 1 \text{ cubic foot}\end{array}\right\} = 12' = 144'' = 1728''' = 20736^{\text{iv}} = \text{etc.}$$

194. We can easily convert quantities expressed in duodecimals to those expressed in feet and inches, and conversely; remembering that in linear measure the *inch* is the same as the *prime*, in square measure, as the *second*, and in cubic measure, as the *third.*

Example 1. 2 ft. $3'.\ 4'' = 2$ ft. $3'\frac{4}{12} = 2$ ft. $3\frac{1}{3}$ in.

Example 2. 3 sq. ft. $2'.\ 4''.\ 3''' = 3$ sq. ft. $28''\frac{3}{12} = 3$ sq. ft. $28\frac{1}{4}$ in.

Example 3. 7 cu. ft. $1'.\ 2''.\ 5'''.\ 6^{\text{iv}} = 7$ cu. ft. $173'''\frac{6}{12}$
$= 7$ cu. ft. $173\frac{1}{2}$ in.

Conversely,

Example 4. 4 yd. 3 ft. $2\frac{1}{3}$ in. $= 15$ ft. $2'\frac{1}{3} = 15$ ft. $2'.\ 4''$.

Example 5. 2 sq. ft. $19\frac{2}{3}$ in. $=$ 2 sq. ft. $19''\frac{2}{3} =$ 2 sq ft. $1'.7''.8'''$.

Example 6. 11 cu. ft. $1000\frac{1}{4}$ in. $=$ 11 cu. ft. $1000'''\frac{1}{4}$
$=$ 11 cu. ft. $83''.4'''\frac{1}{4} =$ 11 cu. ft. $6'.11''.4'''.3^{\text{iv}}$.

EXAMPLES. 120.

Express in yards, feet and inches :

1. 12 ft. $7'.5''$.
2. 20 ft. $8'.3''.9'''$.
3. 13 sq. ft. $9'.3''$.
4. 22 sq. ft. $3'.4''.8'''$.
5. 40 sq. ft. $1'.0''.3'''$.
6. 2 sq. ft. $2'.2''.2'''.2^{\text{iv}}$.
7. 30 cu. ft. $3'.4''$.
8. 74 cu. ft. $7'.3''.4'''$.
9. 10 cu. ft. $2'.1''.0'''.4^{\text{iv}}$.
10. 3 cu. ft. $3'.3''.3'''.3^{\text{iv}}.3^{\text{v}}$.

Express in duodecimals :

11. 2 yd. 2 ft. 7 in.
12. 11 yd. 1 ft. $7\frac{1}{2}$ in.
13. 8 ft. $11\frac{5}{9}$ in.
14. 10 ft. $9\frac{2}{5}$ in.
15. 6 sq. yd. 2 ft. $71\frac{1}{2}$ in.
16. 7 sq. yd. 7 ft. $60\frac{3}{8}$ in.
17. 2 cu. yd. 8 ft. $150\frac{2}{3}$ in.
18. 1 cu. yd. 1 ft. $240\frac{1}{9}$ in.

195. The following statements can be proved as in Art. 189.

Feet	into	primes	give	(supl.)	primes ;
"	"	seconds	"	"	seconds ;
"	"	thirds	"	"	thirds ; etc.
Primes	"	primes	"	"	seconds ;
"	"	seconds	"	"	thirds ; etc.
Seconds	"	seconds	"	"	fourths ;
"	"	thirds	"	"	fifths ; etc.

Also

(Supl.)	feet	into	primes	give	(solid)	primes ;
"	"	"	seconds	"	"	seconds ; etc.
"	primes	"	primes	"	"	seconds ;
"	"	"	seconds	"	"	thirds ; etc.

Example 1. Find the area of a rectangle 7 ft. 8 in. by 6 ft. 7 in.

We multiply all the terms of the multiplicand (commencing with the lowest) by all the terms of the multiplier (commencing with the highest).

ft.	′		
7 .	8		
6 .	7		
46 .	0		$=(7$ ft. $8')\times 6$ ft.
4 .	5	8	$=(7$ ft. $8')\times 7'$.
50 .	5	8	$=(7$ ft. $8')\times(6$ ft. $7')$.

Area $=$ 50 sq. ft. $5'.8'' =$ 50 sq. ft. $68'' =$ 50 sq. ft. 68 in.

Example 2. Find the capacity of a cubical vessel whose edge is 2 ft. 3 in.

$$
\begin{array}{rl}
\text{ft.}\ \ ' & \\
2\ .\ 3 & \\
2\ .\ 3 & \\
\hline
4\ .\ 6 & = (2\text{ ft. }3')\times 2\text{ ft.} \\
6\ .\ 9 & = (2\text{ ft. }3')\times 3'. \\
\hline
5\ .\ 0\ .\ 9 & = (2\text{ ft. }3')\times(2\text{ ft. }3'). \\
2\ .\ 3 & \\
\hline
10\ .\ 1\ .\ 6 & = (5\text{ sq. ft. }0'\ .\ 9'')\times 2\text{ ft.} \\
1\ .\ 3\ .\ 2\ .\ 3 & = (5\text{ sq. ft. }0'\ .\ 9'')\times 3'. \\
\hline
11\ .\ 4\ .\ 8\ .\ 3 & = (5\text{ sq. ft. }0'\ .\ 9'')\times(2\text{ ft. }3').
\end{array}
$$

$\therefore$ Capacity $=11$ cu. ft. $4'\ .\ 8''\ .\ 3''' = 11$ cu. ft. $675''' = 11$ cu. ft. 675 in.

EXAMPLES. 121.

Find by Cross Multiplication the areas of the following rectangles :

1. 3 ft. 4 in. by 2 ft. 3 in.
2. 8 ft. 9 in. by 7 ft. 8 in.
3. 12 ft. 9 in. by 10 ft. 5 in.
4. 16 ft. 11 in. by 12 ft. 10 in.
5. 20 ft. $7\frac{1}{2}$ in. by 15 ft. 4 in.
6. 40 ft. 6 in. by 3 ft. $2\frac{1}{8}$ in.
7. 13 ft. $8\frac{5}{9}$ in. by 7 ft. $2\frac{1}{4}$ in.
8. 12 ft. $9\frac{3}{4}$ in. by 10 ft. $2\frac{2}{3}$ in.
9. 24 ft. $6\frac{7}{8}$ in. by 9 ft. $3\frac{5}{8}$ in.
10. 120 ft. $3\frac{1}{9}$ in. by 20 ft. $5\frac{4}{9}$ in.

Find the volumes of the following rectangular solids :

11. 4 ft. 7 in. by 3 ft. 9 in. by 2 ft. 3 in.
12. 6 ft. 8 in. by 5 ft. 7 in. by 3 ft. 5 in.
13. 10 ft. $8\frac{3}{4}$ in. by 9 ft. 6 in. by 8 ft. 7 in.
14. 12 ft. $3\frac{2}{3}$ in. by 7 ft. $4\frac{1}{4}$ in. by 5 ft. $2\frac{1}{2}$ in.
15. 20 ft. $7\frac{5}{9}$ in. by 15 ft. $8\frac{3}{8}$ in. by 10 ft. $2\frac{5}{8}$ in.

N. B. For additional examples, see the two preceding sections.

XXXV. PROBLEMS AND THE UNITARY METHOD.

196. When the value, weight or length, etc., of any number of units is given, we can, by *Compound Division*, obtain the value, weight or length, etc., of one of the units. And when the value, weight or length, etc., of one unit is given, we can, by *Compound Multiplication*, obtain the value, weight or length, etc., of any number of units of the same kind.

The solution by the application of the two above principles is called the **Unitary Method** or the **Method of Reduction to the Unit**. The method will be fully explained by the following examples.

197. *Example* 1. If 9 articles cost ₨36, what is the cost of 1 article?

The cost of 9 articles = ₨36,

$\therefore$ .... 1 article = ₨$\frac{36}{9}$

= ₨4. *Ans.*

Example 2. If 1 lb. of tea costs 2*s*. 6*d*., what will 8 lb. cost?

The cost of 1 lb. = 2*s*. 6*d*.,

$\therefore$ .... 8 ... = (2*s*. 6*d*.) × 8

= £1. *Ans.*

EXAMPLES. 122.

1. If 7 articles cost ₨2. 10*a*., what is the cost of 1 article?

2. If 12 maunds of wheat cost ₨30, what will 1 maund cost?

3. If $7\frac{1}{2}$ yards of cloth cost ₨1. 14*a*., how much will 1 yd. cost?

4. If the weight of 16 equal bags of rice be 40 maunds, what is the weight of 1 bag?

5. If the length of a piece of cloth worth 18*s*. be 12 yards, what is the length of a piece of the same cloth worth 1*s*.?

6. If the rent of 13 acres of land is £4. 17*s*., what is the rent of 1 acre?

7. If the income-tax on ₨200 be ₨5 . 3 . 4, what is the tax on ₨1?

8. If a chair costs ₨2. 12*a*., how much will 13 chairs cost?

9. If 1 lb. of sugar costs 7*d*., what will 10 lb. cost?

10. If 1 bullock can plough $3\frac{1}{5}$ bighas in a day, how many bighas can 11 bullocks plough in a day?

11. If a man walk $3\frac{3}{4}$ miles in 1 hour, how far does he walk in $9\frac{1}{4}$ hours?

12. A servant's wages being 7*s*. 6*d*. per week, how much ought he to receive for 7 weeks?

13. If the railway fare for 1 mile is $2\frac{1}{2}$*p*., what is the fare for 24 miles?

14. If the carriage of 1 maund for 150 miles cost ₨2, what will be the cost of the carriage of $10\frac{1}{2}$ maunds for the same distance?

Example 3. If 5 men can do a piece of work in 3 days, how long will it take 1 man to do it?

5 men can do the work in 3 days,

∴ 1 man .... (3×5) days,

i.e., 15 days. *Ans.*

Example 4. If 1 man can do a piece of work in 21 days, in how many days can 3 men do it?

1 man can do the work in 21 days,

∴ 3 men .... $\frac{21}{3}$ days,

i.e., 7 days. *Ans.*

Note. In questions such as the two above, it should be noticed that to an **increase** in the number of workmen corresponds a **diminution** in the number of days, and *vice versa.*

EXAMPLES. 123.

1. If 10 men can do a piece of work in 3 days, how long will it take one man to do it?

2. If 12 men finish a piece of work in 5 days, in how many days could one man finish it?

3. If 3 maunds of rice last 9 persons 30 days, how long would they last 1 person?

4. If 7 cwt. can be carried 100 miles for 3*s*., how far can 1 cwt. be carried for the same sum?

5. If 13 acres can be rented for 7 months for a certain sum, for how many months can 1 acre be rented for the same sum?

6. If 1 man can do a piece of work in $40\frac{1}{2}$ days, how long will it take 9 men to do it?

7. If 30 bushels feed 28 horses for a week, how many horses would they keep for 4 weeks?

8. If 1 man reap a field in 18 days, how long will 4 men be doing it?

9. A ship performs a voyage in 55 days, sailing 1 knot an hour, how many days would she take to perform the same voyage sailing 5 knots an hour?

10. If the carriages of 56 maunds for 1 mile cost a certain sum, how much will be carried 14 miles for the same money?

11. If 18 horses plough a field in 15 days, how many horses will plough it in 1 day?

But B A C, D C A are the angles of incidence and refraction. Thus the two laws of refraction are obeyed.

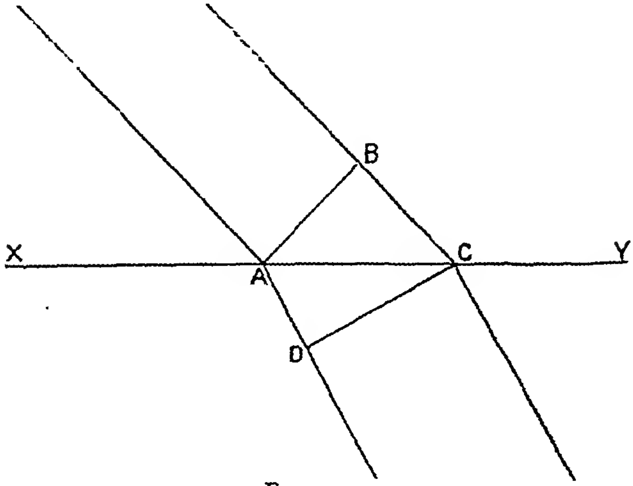


FIG. 140.

We have now a crucial test between the emission and the undulatory theory. In passing from one medium to another in which the velocity of light is less, according to the emission theory the index of refraction should be less than unity; according to the undulatory theory it should be greater than unity. Now, direct experiments on the velocity of light in various media, and on refractive indices, show that it is the undulatory theory that gives the correct explanation.

Rectilinear Propagation of Light.—The emission theory readily accounts for the rectilinear propagation of light. This was one of the greatest difficulties in the way of the undulatory theory. But this theory can satisfactorily, although not easily, account for this behaviour of light.

Let O be a visible point radiating disturbances into

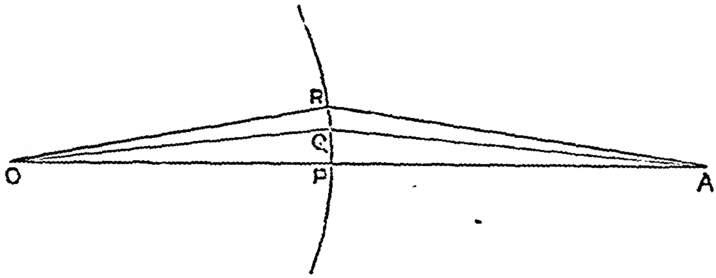


FIG. 141.

surrounding space. Let A be a point at a distance a from O. Suppose a series of periodic waves of length λ to be sent out

Example 5. Express 1 mile in metres, 32 metres being equal to 35 yards.

$$35 \text{ yards} = 32 \text{ metres},$$
$$\therefore \quad 5 \text{ yards} = \tfrac{32}{7} \text{ metres},$$
$$\therefore \quad 1760 \text{ yards} = \tfrac{32 \times 352}{7} \text{ metres or } 1609\tfrac{1}{7} \text{ metres.}$$

EXAMPLES. 124.

1. If 30 bullocks cost ₨810, what is the cost of 77 bullocks?

2. If 5 cwt. cost ₨6. 4*a.*, what is the cost of 16 cwt.?

3. Find the value of 21 yd. of cloth when 44 yd. cost ₨33.

4. If 7 pieces of cloth cost ₨350, what will 13 pieces cost?

5. If 13 reams of paper cost £6. 10*s.*, what is the price of 21 reams?

6. If 23 copies of a book cost ₨35. 15*a.*, how much will 31 copies cost?

7. If the cost of 60 eggs be 1*s.* 3*d.*, how many can be purchased for 5*s.*?

8. How many oranges can be bought for ₨2. 3*a.* at the rate of 8*a.* 9*p.* a dozen?

9. If 4 cwt. cost £1. 1*s.* 1*d.*, what will 2 tons 8 cwt. cost?

10. If 35 sheep produce 20 lb. of wool, what would 63 sheep produce?

11. If 42 men earn ₨3 . 4 . 6 for a day's work, what would 112 men earn?

12. If the railway fare for 100 miles be ₨3 . 8 . 6, what is the fare for 275 miles?

13. If 8 persons can be boarded for £3, how many can be boarded for £7. 10*s.*?

14. What is the value of 600 pins at the rate of 2*d.* per gross?

15. If $7\frac{3}{4}$ lb. cost 2*s.* 7*d.*, what will $1\frac{1}{4}$ cwt. cost?

16. If $\frac{3}{4}$ of a maund cost ₨3. 12*a.*, find the cost of $3\frac{3}{4}$ seers.

17. If $\frac{3}{7}$ of an estate be worth ₨2700, what is the value of $\frac{5}{9}$ of the estate?

18. If $\frac{7}{24}$ of a cargo be worth £357. 7*s.*, what is the value of $\frac{2}{3}$ of the cargo?

19. The owner of ·375 of a ship sold $\frac{3}{8}$ of his share for ₨5040; find the value of ·875 of the ship at the same rate.

20. A man lost $\frac{2}{5}$ of his money, and then spent $\frac{3}{4}$ of the remainder; after which he had ₨120 left: how much did he lose?

21. A gentleman possessing $\frac{3}{14}$ of an estate sold $\frac{2}{7}$ of $\frac{5}{16}$ of his share for R241. 4*a.*; what would ·2 of $\frac{3}{16}$ of the estate sell for at the same rate?

22. If a man walk 46 miles in 3 days, in how many days will he walk 115 miles?

23. If the rent of 34 acres is R21. 4*a.*, what is the rent of 51 acres?

24. A servant's wages being £10. 8*s.* per annum, how much ought she to receive for 7 weeks? [1 year = 52 weeks.]

25. A man's annual income is R4088: what does he receive for 15 days? [1 year = 365 days.]

26. If 27 bus. $2\frac{1}{2}$ pk. cost £10 . 7 . $2\frac{1}{4}$, what is the cost of a bushel and a half?

27. If 3 cwt. 3 qr. cost £6. 15*s.*, what will be the cost of 2 cwt.?

28. A sack of potatoes weighs 89 seers; if 6 such sacks cost R22. 4*a.*, what will be the cost of 22 seers?

29. If 17 ac. 2 ro. 38 po. supply 3 horses, how many acres will supply 16 horses?

30. If the carriage of 25 maunds for 500 miles cost R9. 6*a.*, what weight can be carried the same distance for R8?

31. If a piece of land worth R375 yield an income of R7. 8*a.*, what should be the value of a piece of land which yields an income of R18. 12*a.*?

32. If $3\frac{2}{3}$ acres can be mown in 7 days, how long will it take to mow $9\frac{1}{3}$ acres?

33. If 350 rupees weigh 9 lb., how many pounds will 625 rupees weigh?

34. In a certain time the population of a town increased from 78960 to 82908; find by how many the population of another town of 92360 inhabitants would have increased at the same rate in the same time.

35. A man walks 4 miles in an hour; how many yards does he walk in a minute.

36. A railway train travels at the rate of 20 miles in $1\frac{1}{2}$ hours; find the rate per minute.

37. An express train goes 10 times as fast as a man who walks 6 ft. in a second: how many miles per hour does it go?

38. Express $7\frac{1}{2}$ miles in kilometres, 5 kilometres being equal to 5456 yards.

39. If $6\frac{1}{2}$ grammes be equal to 105 grains, express a pound avoir. in grammes.

40. Convert £3 . 7 . 6 to Indian money, given ₨8 = 15s.

41. Convert 7 tons to maunds, given 35 seers = 72 lb.

42. Express $3\frac{1}{8}$ dollars in Indian money, 9 dollars being equal to 20 rupees.

43. If 8 horses eat as much as 6 oxen, how many oxen will eat as much as 20 horses.

44. If 4 men do as much work as 6 boys, how many men will do the work of 18 boys?

45. If the price of 7 horses and 5 oxen is ₨520, and that of an ox is ₨20, find the price of a horse.

46. If the weight of 5 rupees and 3 pice is 1200 grains, and that of a rupee is 180 grains, find the weight of a pice.

47. If 8 horses and 20 sheep eat the grass of 7 acres in a certain time, how many acres will feed 10 horses and 24 sheep for the same time, supposing a horse to eat as much as 4 sheep?

48. If 15 chairs and 2 tables cost ₨400, find the cost of 12 chairs and 3 tables, the cost of 10 chairs being equal to that of 4 tables.

49. If the wages of 4 men be equal to those of 5 women, what will 8 women earn in a day, the daily earnings of 10 men being ₨1. 9*a.*?

50. If a shop-keeper uses a weight of 15 oz. for 1 lb., how much will a customer lose in buying 24 lb.?

Example 6. If 35 men finish a piece of work in 8 days, how many men will finish it in 10 days?

In 8 days the work is done by 35 men,

∴ ... 2 .... 35×4 ......,

∴ ...10 .... $\frac{35 \times 4}{5}$ ......,

or 28 men. *Ans.*

Example 7. If the penny loaf weighs 12 oz. when wheat is £4 a quarter, what should it weigh when wheat is £4. 16*s.* a quarter?

£4 = 80*s.*; £4. 16*s.* = 96*s.*

When wheat is 80*s.* a qr. the loaf weighs 12 oz.,

∴ .... 16*s.* .... (12×5) oz.,

∴ .... 96*s.* .... $\frac{12 \times 5}{6}$ oz.,

or 10 oz. *Ans.*

Example 8. A garrison of 1200 men is provisioned for 60 days; if after 15 days 300 men leave the garrison, how long will the remaining provisions last the men left?

The provisions left would last 1200 men 45 days,

∴ they would last 300 men (45×4) days,

∴ they would last 900 men $\frac{45 \times 4}{3}$ days, or 60 days. *Ans.*

EXAMPLES. 125.

1. If 9 men can mow a field in 4 days, in how many days could 6 men mow the same field?

2. If 12 horses can plough a field in 7 days, in how many days could 14 horses plough it?

3. If 16 men finish a piece of work in 5 days, in how many days could 10 men do it?

4. If 25 men reap a field in 12 days, how many men could reap it in 20 days?

5. If 7 cwt. feed 15 horses for 8 days, how many horses would they feed 12 days?

6. If 28 maunds can be carried 50 miles for a certain sum, what weight can be carried 125 miles for the same sum?

7. If 16 bighas can be rented for 9 months for R10, for how many months can 36 bighas be rented for the same sum?

8. A man walks from Calcutta to Hugly in 6 hours, walking 4 miles an hour; how long would he take if he rode at the rate of 9 miles an hour?

9. If the twopenny loaf weighs 20 oz. when wheat is £4. 16*s*. a quarter, what should it weigh when wheat is £8 a quarter?

10. If the sixpenny loaf weighs 64 oz. when wheat is 6*s*. 9*d*. a bushel, what is the price of wheat per bushel when the sixpenny loaf weighs 48 oz.?

11. From a mass of silver I can make 64 plates weighing 3 oz. each, how many 4 oz. plates could I make from the same?

12. A garrison of 1200 men has provisions for 75 days; how long would they last if the garrison were reduced to 500 men?

13. A fortress is provisioned for 4 weeks at the rate of 20 oz. a day for each man: if only 12 oz. be served out daily for each man, how long can the place hold out?

14. A garrison of 1000 men is provisioned for 70 days: if after 20 days the garrison is re-enforced by 200 men, how long will the remaining provisions last?

15. If 7 men can mow a meadow in 7 days, working 10 hours

a day, how many additional hours a day must they work to do it in 5 days?

16. If I borrow ₨300 for 8 months, for how long should I lend ₨400 in return?

17. If it requires $27\frac{1}{2}$ yd. of carpet 9 in. wide to cover a room, how many yards of carpet 7 in. wide will be necessary to cover the same room?

EXAMPLES. 126.

1. If 30 seers of corn feed 6 horses for 4 days, how many horses would they feed for 12 days?

2. If 30 seers of corn feed 6 horses for 4 days, how many horses would 25 seers feed for the same time?

3. If 30 seers of corn feed 6 horses for 4 days, for how many days would they feed 8 horses?

4. If 30 seers of corn feed 6 horses for 4 days, for how many days would $52\frac{1}{2}$ seers feed the same number of horses?

5. If 30 seers of corn feed 6 horses for 4 days, how many seers will feed 10 horses for the same time?

6. If 30 seers of corn feed 6 horses for 4 days, how many seers will feed the same number of horses for 9 days?

7. If 20 men reap a field of 6 acres in 40 hours, in how many hours will 35 men reap the same field?

8. If 20 men reap a field of 6 acres in 40 hours, how many men will reap the same field in 25 hours?

9. If 20 men reap a field of 6 acres in 40 hours, how many acres will 35 men reap in the same time?

10. If 20 men reap a field of 6 acres in 40 hours, how many men will reap 15 acres in the same time?

11. If 20 men reap a field of 6 acres in 40 hours, how many acres will they reap in 55 hours?

12. If 20 men reap a field of 6 acres in 40 hours, in how many hours will they reap a field of 8 acres?

13. When rice is ₨3 per md., how many people can be fed for the same sum that would feed 90 people when rice is ₨2. 8*a.* per md.?

14. If 1 lb. of flour cost 9*p.* when wheat is ₨3 per md., what should be the price of a md. of wheat when 1 lb. of flour costs 1*a.*?

15. How many yards of cloth worth 4*a.* 6*p.* per yard must be given in exchange for 30 yards at 3*a.* 6*p.* per yard?

16. Find the length of a strip of land 20 yd. wide, that should be given in exchange for a piece measuring 40 yd. by 30 yd.

17. If 3 lb. of tea cost as much as 10 lb. of sugar, how much tea should be given in exchange for 25 lb. of sugar?

18. A brewer receives 10 doz. of brandy in exchange for 4 barrels of ale worth £3. 10*s.* a barrel; what does the brandy cost him per bottle?

19. A man contracts to perform a piece of work in 20 days and immediately employs upon it 16 men. At the end of 12 days the work is only half done; what additional number of men must he employ to fulfil the contract?

20. A merchant of Calcutta indented from London goods worth £640, and paid £10 for freight. If a rupee is equal to 1*s.* 9*d.*, for how many annas must he sell goods, for which he paid 1*s.* to the London manufacturer, in order to gain £50 on the whole outlay?

21. If a quantity of flour serve 36 men for 15 days at the rate of 12 oz. a day for each man, how many ounces a day will each man get, when the same quantity of flour serves 42 men for the same time.

22. When grain is R2 per md. how many horses can be kept for the same sum that would keep 20 horses when grain is R1. 8*a* per md.?

Example 9. If 10 men can do a piece of work in 12 days, working 7 hours a day, how many hours a day must 6 men work to do the same in 14 days?

10 men can do the work in (12×7) hours,

$\therefore$ 2 .... $(12\times7\times5)$ ....,

$\therefore$ 6 .... $\frac{12\times7\times5}{3}$ ....;

$\therefore$ to complete the work in 14 days, they must work $\frac{12\times7\times5}{3\times14}$ hours, or 10 hours a day.

Example 10. If a number of men can dig a trench 210 yd. long, 3 wide and 2 deep, in 5 days of 11 hours each, in how many days of 10 hours each, will they dig a trench 420 yd. long, 6 wide and 3 deep?

$(210\times3\times2)$ cu. yd. is dug in 55 hours.

$\therefore$ 1 .... $\frac{55}{210\times3\times2}$ hours,

$\therefore$ $(420\times6\times3)$ .... $\frac{55\times420\times6\times3}{210\times3\times2}$ hours,

or 330 hours;

$\therefore$ the number of days required $=\frac{330}{10}=33$.

Example 11. If 8 oxen or 6 horses eat the grass of a field in 10 days, in how many days will 5 oxen and 4 horses eat it?

8 oxen eat as much as 6 horses,

∴ 1 ox eats .... $\frac{6}{8}$ horses,

∴ 5 oxen eat .... $\frac{6\times5}{8}$ horses, or $\frac{15}{4}$ horses;

∴ 5 oxen and 4 horses eat as much as $(\frac{15}{4}+4)$ horses or $\frac{31}{4}$ horses.

Now, 6 horses eat the grass in 10 days,

∴ 1 horse will eat .... 10×6 ......,

∴ $\frac{31}{4}$ horses .... $\frac{10\times6\times4}{31}$ ......,

or $7\frac{23}{31}$ days.

EXAMPLES. 127.

1. If 5 men can do a piece of work in 8 days, working 7 hours a day, how many men will do the same piece of work in $4\frac{2}{3}$ days, working 10 hours a day?

2. If 9 men can do a piece of work in 7 days, working 10 hours a day, how many hours a day must 6 men work to do the same in 30 days?

3. If 12 men can do a piece of work in 8 days of 7 hours each, in how many days of 6 hours each can 10 men do the same?

4. If 20 masons build a wall, 50 ft. long, 2 ft. thick and 14 ft. high, in 12 days, in how many days will they build a wall, 55 ft. long, 4 thick and 16 high?

5. If 20 men dig a trench, 100 yd. long, 5 wide and 3 deep, in 3 days, how many men will dig a trench 150 yd. long, 6 wide and 2 deep, in the same time?

6. If 5 men reap a rectangular field, 200 ft. by 50 ft., in 2 days of 10 hours each, in how many days of 8 hours each can they reap another, 300 ft. by 40 ft.?

7. If 6 men or 8 boys can do a piece of work in 18 days; in how many days will 3 men and 5 boys do it?

8. If 5 men, 7 women or 9 boys can dig a ditch in 15 days, in how many days can 1 man, 1 woman and 1 boy dig it?

9. 4 men do as much work as 6 boys in the same time, and a piece of work in which 20 men and 15 boys are engaged takes 25 days; how many days would it take if 15 men and 20 boys were employed upon it?

10. If 10 gas-burners, which are lighted 4 hours every evening for 15 days, consume a quantity of gas which costs ₨3, for how many days can 12 burners be lighted 5 hours every evening at the same cost?

11. If a piece of matting, measuring 7 ft. 4 in. by 5 ft., cost R6. 14*a*., what will be the cost of a piece of the same matting, measuring 10 ft. by 6 ft. 6 in.?

12. If the cost of printing a book of 250 pages, with 21 lines on each page, and on an average 10 words in each line, be R125, find the cost of printing a book of 300 pages, with 14 lines on each page and 8 words in each line.

13. If 8 men, working 7 hours a day, take 12 days to complete a piece of work, how long will 14 boys, working 6 hours a day, take to do the same work, the work of one man being equal to that of two boys in the same time?

14. If the feeding of 8 horses and 20 sheep for a month cost R100, what will be the cost of feeding 6 horses and 50 sheep for a month, supposing that 2 horses eat as much as 15 sheep?

BANKRUPTCIES, RATING, TAXING, ETC.

199. *Example* 1. A bankrupt's debts are R7240, and his assets (*i.e.*, the value of his property) are R5430; how much can he pay in the rupee?

In the place of R7240 he can pay R5430,

$\therefore$ ....R1 .... R$\frac{5430}{7240}$, or R$\frac{3}{4}$,
or 12 annas;

$\therefore$ he can pay 12*a*. in the rupee.

Example 2. A bankrupt's debts amount to £3720, and he pays 18*s*. in the pound; what are his assets?

In the place of £1 he pays 18*s*.,

$\therefore$ .... £3720 .... (3720 × 18)*s*.,

$\therefore$ his assets are (3720 × 18)*s*., or £3348.

Example 3. A man pays an income-tax of R125 at the rate of 5*p*. in the rupee; find his income.

R125 = 24000*p*.

He pays 5*p*. on R1,

$\therefore$ .... 24000*p*. on R4800;

$\therefore$ his income is R4800.

Example 4. After paying an income-tax of 6*d*. in the pound a man has £780 left; find his gross income.

He has 19*s*. 6*d*. left out of £1,

$\therefore$ .... 1*s*. .... £$\frac{2}{39}$;

$\therefore$ .... (780 × 20)*s*. .... £$\frac{2 \times 780 \times 20}{39}$, or £800;

$\therefore$ his gross income is £800.

Example 5. A man pays an income-tax of 6*p.* in the rupee on $\frac{2}{3}$ of his income; how much in the rupee does he pay on his whole income?

He pays 6*p.* in the rupee on $\frac{2}{3}$ of his income, *i.e.*, he pays $\frac{6}{16 \times 12}$ of $\frac{2}{3}$ of his income, or $\frac{1}{48}$ of his income. But $\frac{1}{48}$ of R1 = 4*p.*; ∴ he pays 4*p. in the rupee on his whole income.*

Example 6. When income-tax is 5*p.* in the rupee a person has to pay R20 more than when the tax was 4*p.* in the rupee; find his income.

Difference of tax is 1*p.* when the income is R1.

∴ .... (20 × 16 × 12)*p.* .... R(20 × 16 × 12),

or R3840;

∴ his income is R3840.

EXAMPLES. 128.

1. Find the income-tax on R3600 at 5*p.* in the R.

2. How much will a poor-rate of 2*s.* 6*d.* in the £ produce in a parish where the whole property is rated at £3768. 8*s.*?

3. Find the amount of road-cess, at 6*p.* in the R, on a rental of R5500.

4. A bankrupt's debts are R7880, and his assets R4925; how much in the rupee can he pay?

5. A bankrupt's effects amount to R6131 . 5 . 4, and his debts are R36788; how much can he pay in the rupee?

6. If a man has to pay £9 . 7 . 6 for income-tax on an income of £750, what is the rate of tax per £?

7. A bankrupt's debts are R3798, and he pays 12*a.* 6*p.* in the rupee; what are his assets?

8. A bankrupt's assets are £2900, and he pays his creditors 14*s.* 6*d.* in the £; what do his debts amount to?

9. A man pays an income-tax of R40 at the rate of 4*p.* in the rupee; find his income.

10. If I pay £16. 10*s.* 6*d.* for income-tax, being at the rate of 10*d.* in the £, what is my income?

11. After paying an income-tax of 5*p.* in the rupee a man has R2805 left; find his gross income.

12. A person after paying 7*d.* in the £ for income-ta £174. 15*s.* left; what was his gross income?

13. A creditor received 16*s.* 3*d.* in the £, and th £135. 10*s.*; how much was due to him?

14. A man pays an income-tax of $4p.$ in the rupee on $\frac{7}{8}$ of his income; what rate per rupee does he pay on his whole income?

15. A man pays an income-tax of $8p.$ in the rupee on $\frac{3}{5}$ of his income; what fraction of his whole income is paid as income-tax?

16. When the income-tax is $9d.$ in the pound a person has to pay £40 less than when the tax was $1s.$ in the pound; find his income.

17. When the income-tax is $7d.$ in the pound a person has to pay £25 more than when the tax was $5d.$ in the pound; find his income.

PROBLEMS RELATING TO WORK DONE IN A CERTAIN TIME.

200. *Example* 1. *A* can do a piece of work in 7 days, and *B* can do it in 9 days; how long will *A* and *B*, working together, take to do the work?

A can do the work in 7 days, $\therefore$ *A* can do $\frac{1}{7}$ of it in 1 day;
B .... 9 ......, $\therefore$ *B* .... $\frac{1}{9}$ ....;
$\therefore$ *A* and *B* together can do $(\frac{1}{7}+\frac{1}{9})$ of it in one day,
$\therefore$ .... $\frac{16}{63}$ ....,
$\therefore$ .... the whole in $\frac{63}{16}$ days;
$\therefore$ the time required $=\frac{63}{16}$ days $=3\frac{15}{16}$ days.

Example 2. *A* and *B* together can perform a piece of work in 5 days, and *A* alone can do it in 8 days: what time will it take *B* to do it alone?

A and *B* can do the work in 5 days, $\therefore$ they can do $\frac{1}{5}$ of it in 1 day;
A alone .... 8 ......, $\therefore$ he .... $\frac{1}{8}$ ....;
$\therefore$ *B* alone can do $(\frac{1}{5}-\frac{1}{8})$ of it in 1 day,
$\therefore$ .... $\frac{3}{40}$ ....,
$\therefore$ .... the whole in $\frac{40}{3}$ days or $13\frac{1}{3}$ days. *Ans.*

Example 3. A vessel can be filled by a pipe in 25 minutes, and it can be emptied by a waste pipe in 20 minutes: if both the pipes be opened when the vessel is full, how soon will it be empty?

1st pipe fills $\frac{1}{25}$ of the vessel in 1 minute,
2nd pipe empties $\frac{1}{20}$ ....;
$\therefore$ when both pipes are open
$(\frac{1}{20}-\frac{1}{25})$ of the vessel is emptied in 1 minute,
i.e., $\frac{1}{100}$ ....,
$\therefore$ the whole will be emptied in 100 minutes.

Example 4. A and B can do a piece of work in 5 hours ; A and C in 4 hours ; B and C in $3\frac{1}{2}$ hours. In what time can A alone do it ?

A and B can do $\frac{1}{5}$ in 1 hour ;

A and C .... $\frac{1}{4}$ .... ;

$\therefore$ two men of A's strength, and B and C can do $\frac{1}{5}+\frac{1}{4}$ in 1 hour ,

but B and C can do $\frac{2}{7}$ in 1 hour ;

$\therefore$ two men of A's strength can do $\frac{1}{5}+\frac{1}{4}-\frac{2}{7}$ in 1 hour,

or $\frac{23}{140}$ in 1 hour ;

$\therefore$ A can do $\frac{23}{280}$ in 1 hour ;

$\therefore$ A can do the whole in $\frac{280}{23}$ hours, or $12\frac{4}{23}$ hours. *Ans.*

Example 5. A does $\frac{4}{5}$ of a piece of work in 20 days ; he then calls in B, and they finish the work in 3 days ; how long would B take to do the whole work by himself ?

	A does	$\frac{4}{5}$	of the work in 20 days,
$\therefore$	A can do	$\frac{1}{25}$	of the work in 1 day,
$\therefore$	A does	$\frac{3}{25}$	of the work in 3 days,
	but A and B do	$\frac{1}{5}$	of the work in 3 days,
$\therefore$	B does	$(\frac{1}{5}-\frac{3}{25})$	of the work in 3 days,
	i.e., ...	$\frac{2}{25}$	............................,
$\therefore$	B can do	$\frac{2}{75}$	......................1 day,

$\therefore$ B can do the whole work in $\frac{75}{2}$ days, or $37\frac{1}{2}$ days. *Ans.*

EXAMPLES. 129.

1. A can do a piece of work in 10 hours ; B can do it in 8 hours. In what time will they do it if they work together ?

2. If A does a piece of work in 4 days, which B can do in 5, and C can do in 6, in what time will they do it, all working together ?

3. A cistern can be filled by one pipe in $3\frac{1}{2}$ hours, by a second in $3\frac{1}{3}$ hours, and by a third in $5\frac{1}{4}$ hours ; in what time will it be filled by all the three in action together ?

4. A can reap a field in 10 days ; B can reap it in 12 days ; C can reap it in 15 days ; how long will it take them all together to reap it, and what part of the work will be done by each ?

5. A and B together can dig a trench in 4 days, and A alone can dig it in 6 days ; in how many days can B alone dig it ?

6. Two pipes, P and Q, together can fill a cistern in 20 minutes, and P alone in 30 minutes : how long would Q alone take ?

7. A vessel can be filled by one pipe in 8 minutes, by a second pipe in 10 minutes; it can be emptied by a waste pipe in 12 minutes: in what time will the vessel be filled if all the three be opened at once?

8. A vessel has 3 pipes connected with it, 2 to supply and 1 to draw off. The first alone can fill the vessel in $4\frac{1}{2}$ hours, the second in 3 hours, and the third can empty it in $1\frac{1}{2}$ hours. If all the pipes be opened when the vessel is half-full, how soon will it be empty?

9. *A* and *B* can do a piece of work in 6 days; *A* and *C* in $5\frac{1}{2}$ days; *B* and *C* in 4 days. In what time could each do it?

10. *A* and *B* can mow a field in $3\frac{1}{2}$ days; *A* and *C* in 4 days; *B* and *C* in 5 days. In what time could they mow it, all working together?

11. *A* does $\frac{2}{5}$ of a piece of work in 9 days; he then calls in *B*, and they finish the work in 6 days. How long would *B* take to do the whole work by himself?

12. *A* does $\frac{7}{10}$ of a piece of work in 15 days; he does the remainder with the assistance of *B* in 4 days. In what time could *A* and *B* together do it?

13. *A* can do a piece of work in 16 days, *B* in 10 days; *A* and *B* work at it together for 6 days, and then *C* finishes it in 3 days: in how many days could *C* have done it alone?

14. *A* and *B* together can do a piece of work in 6 days, *B* alone could do it in 16 days. If *B* stops after 3 days, how long afterwards will *A* have finished the work?

15. *A* and *B* can reap a field in 30 days, working together. After 11 days, however, *B* is called off, and *A* finishes it by himself in 38 days more. In what time could each alone do the whole?

16. *A*, *B* and *C* together can do a piece of work in 6 days, which *B* alone can do in 16 days, and *B* and *C* together can do in 10 days; in how many days can *A* and *B* together do it?

17. Five men can do a piece of work in 2 hours, which 7 women could do in 3 hours, or 9 children in 4 hours. How long would 1 man, 1 woman and 1 child together take to do the work?

18. *A* can do a piece of work in 4 hours, *B* and *C* can do it in 3 hours, *A* and *C* can do it in 2 hours. How long would *B* alone take to do it?

19. *A* and *B* together can do a piece of work in 8 days; *B* alone can do it in 12 days; supposing *B* alone works at it for 4 days, in how many more days could *A* alone finish it?

20. Three taps, *A*, *B* and *C*, can fill a cistern in 10 min., 12 min. and 15 min. respectively. They are all turned on at once,

but after $1\frac{1}{2}$ min. B and C are turned off. How many minutes longer will A take then to fill the cistern?

21. Two pipes, A and B, can fill a cistern in 3 hours and 4 hours respectively; a waste pipe C can empty it in 2 hours; if these pipes be opened in order at 7, 8 and 9 o'clock, find when the cistern will be filled.

22. A piece of work was to be completed in 40 days; a number of men employed upon it did only half the work in 24 days; 16 more men were then set on, and the work was completed in the specified time: how many men were employed at first?

23. A can do a certain work in the same time in which B and C together can do it. If A and B together could do it in 10 days, and C alone in 50 days, in what time could B alone do it.?

24. A and B can do a piece of work in 10 days, B and C in 15 days, and A and C in 25 days: they all work at it together for 4 days; A then leaves, and B and C go on together for 5 days more, and then B leaves: in how many more days will C complete the work?

25. A cistern can be filled by two pipes in 30 and 40 minutes respectively; both the pipes were opened at once but after some time the first was shut up, and the cistern was filled in 10 minutes more. How long after the pipes had been opened was the first pipe shut up?

26. A cistern has 3 pipes, A, B and C; A and B can fill it in 2 and 3 hours respectively; C is a waste pipe. If all the three pipes be opened at once $\frac{7}{24}$ of the cistern will be filled up in 30 minutes. In what time can C empty the full cistern?

27. Forty men can finish a piece of work in 40 days; but if 5 men leave the work after every tenth day, in what time will the whole work be completed?

PROBLEMS RELATING TO CLOCKS.

201. *Example* I. Two clocks are at 12 noon; one gains 40 seconds and the other loses 50 seconds in 24 hours: after what interval will the one have gained 16 minutes on the other, and what time will each then show? What will be the true time when the first clock indicates 3 P. M. on the following day?

(i) The one clock gains on the other $(40+50)$ seconds in 24 hours;

i.e., it gains $\frac{3}{2}$ min. in 1 day,

$\therefore$ .... 1 .... $\frac{2}{3}$ .....,

$\therefore$ ....16 ....$\frac{2\times16}{3}$ days, or $\frac{32}{3}$ days,

or 10 days 16 hours (true time).

(ii) Now in $\frac{32}{3}$ days the first clock gains $\frac{32}{3} \times 40$ sec. or $7\frac{1}{9}$ min., and the second loses $\frac{32}{3} \times 50$ sec. or $8\frac{8}{9}$ min.

But the correct clock, at the end of the interval (*i. e.*, 10 days 16 hours) will show 4 A. M.

Therefore the first will show 4 h. $7\frac{1}{9}$ min. A. M.;

and the second will show 3 h. $51\frac{1}{9}$ min. A. M.

(iii) From 12 noon to 3 P. M. on the following day there are 27 hours.

24 h. 40 sec. of the first clock $=$ 1 day of the correct clock,

i.e., $\frac{2161}{90}$ h. .... $=$ 1 day ....,

$\therefore$ 1 h. .... $= \frac{90}{2161}$ da. ....,

$\therefore$ 27 h. .... $= \frac{90 \times 27}{2161}$ da. ....

Now $\frac{90 \times 27}{2161}$ da. $=$ 1 da. 2 h. $59\frac{541}{2161}$ min.

$\therefore$ When the first clock indicates 3 P. M. on the following day, the true time will be 2 h. $59\frac{541}{2161}$ min. P. M.

EXAMPLES. 130.

1. A watch which is 5 minutes too fast at 12 o'clock on Sunday gains 2 min. 15 sec. per day; what time will it indicate at half past 2 P. M. on the following Tuesday?

2. A clock which is 10 minutes too fast at 9 A. M. on Monday loses 3 min. per day; what time will it show at a quarter to 3 P. M. on the following Wednesday?

3. One clock gains 2 minutes, and a second gains 3 minutes in 24 hours: the first is put right at 12 o'clock on Tuesday, the second at 3 P. M. on the following Wednesday: when will they indicate the same time?

4. Two clocks are exactly together at 8 A. M. on a certain day; one loses 6 seconds and the other gains 10 seconds in 24 hours; when will the one be half an hour before the other, and what time will each clock then show?

5. A watch which shows correct time at noon on Tuesday gains $2\frac{1}{2}$ min. a day: what is the correct time on the following Sunday when it is 9 A. M. by the watch?

6. Two clocks strike 9 together on Monday morning; on Tuesday morning one wants 10 minutes to 11, when the other strikes 11. How much must the slower be put on, or the faster put back, that they may strike 9 together in the evening?

7. A clock which was $1 \cdot 4$ min. fast at a quarter to 11 P. M. on Dec. 2, was 8 min. slow at 9 A. M. on Dec. 7; when was it exactly right?

8. A clock which was $1 \cdot 2$ min. fast at a quarter to 11 P. M. on Nov. 28, was exactly right at 11-30 P. M. the following day. How many minutes was it slow at a quarter to 2 P. M. on Dec. 7?

9. A clock which is $7\frac{1}{2}$ min. fast on Tuesday at noon, is $4\frac{1}{4}$ min. fast at midnight on the following Monday; how much did it lose in a day?

10. A watch which gains $7\frac{1}{2}$ min. in a day is 12 minutes fast at midnight on Sunday. What will be the true time when the watch indicates 4-32 P. M. on Wednesday?

11. Two clocks, of which one gains $3\frac{1}{2}$ min. and the other loses $2\frac{1}{2}$ min. in 24 hours, were both within 1 min. of the true time, the former fast and the latter slow, at noon on Sunday last; they now differ from one another by 15 min.: find the day of the week and the hour of the day.

12. A clock loses $2\frac{1}{2}$ minutes a day; how must the hands be placed at 9 A. M. so as to point to true time at noon?

13. One clock gains $12\frac{1}{2}$ minutes, and another gains $7\frac{1}{2}$ minutes in 12 hours. They are set right at noon on Sunday. Determine the time indicated by each clock, when the one appears to have gained $21\frac{2}{3}$ min. on the other.

14. A clock set accurately at 1 o'clock indicates 10 minutes to 6 at 6 o'clock: what is the true time when the clock indicates 6 o'clock?

15. A watch is 73 seconds slow at noon on January 1st 1887: how much must it gain daily that it may be $17\frac{1}{2}$ seconds fast at noon on July 1st?

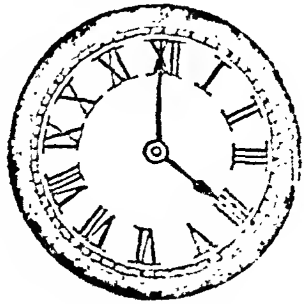
16. A watch is set right at 10 P. M. on Sunday; at 10 A. M. on Wednesday it is 5 minutes too fast; what will be the true time when it is 2 P. M. by the watch on Friday?

17. A watch which gains 5 minutes in 12 hours is put right on January 1st 1888; when will it again show correct time?

18. A church-clock was 15 minutes too fast 10 days ago, and to-day at the same hour it is 15 minutes too slow: when did it show true time? When will it again show true time?

19. Two clocks, of which one gains and the other loses one minute in an hour, strike one o'clock together; what will be the interval, measured by a correct clock, between their respective striking 2?

Example 2. Find the time between 4 and 5 o'clock when the hands of a clock are (i) together, (ii) at right angles, (iii) opposite to each other.



Note. While the minute-hand passes over 60 minute-divisions the hour hand passes over only 5. Therefore in 60 minutes the minute-hand gains 55 divisions on the hour-hand; and therefore in 12 minutes the minute-hand gains 11 divisions on the hour-hand.

At 4 o'clock the minute-hand is 20 divisions behind the other.

(i) The two hands to be together between 4 and 5, the minute-hand has to gain 20 divisions on the hour-hand.

The minute-hand gains 11 divisions in 12 minutes,

∴ .... 1 division in $\frac{12}{11}$ ....,

∴ .... 20 divisions in $\frac{12\times20}{11}$ .... ;

∴ the time required is $\frac{12\times20}{11}$ min. or $21\frac{9}{11}$ min. past 4.

(ii) When the hands are at right angles there is a space of 15 minute-divisions between them. Between 4 and 5 this will happen twice: first, when the minute-hand has gained 5 (*i.e.*, $20-15$) divisions; and secondly, when it has gained 35 (*i.e.*, $20+15$) divisions.

The minute-hand gains 11 divisions in 12 minutes,

∴ .... 1 division in $\frac{12}{11}$ ....,

∴ .... 5 divisions in $\frac{12\times5}{11}$ .... ;

and .... 35 divisions in $\frac{12\times35}{11}$ .... ;

∴ The two hands will be at right angles at $\frac{12\times5}{11}$ min. or $5\frac{5}{11}$ min. past 4; and also at $\frac{12\times35}{11}$ min. or $38\frac{2}{11}$ min. past 4.

(iii) When the hands are opposite to each other, there is a space of 30 divisions between them. This will happen when the minute-hand has gained 50 (*i.e.*, $20+30$) divisions.

The process will be similar to that in the preceding cases. The time is $54\frac{6}{11}$ min. past 4.

EXAMPLES. 131.

At what time are the hands of a clock (i) coincident, (ii) at right angles, (iii) opposite each other, (iv) 12 divisions apart, (v) 22 divisions apart, between the hours of

1. 2 and 3? 2. 3 and 4? 3. 6 and 7?

4. 12 and 1? 5. 7 and 8? 6. 10 and 11?

7. A watch is 10 minutes too fast at noon; it loses 2 min. in one hour: find the true time when its hands are at right angles between 2 and 3 o'clock.

8. A clock is 5 minutes too slow at 1; it gains 1 min. in an hour: what is the true time when its hands are together for the fifth time after 1 o'clock?

9. A clock is put right at 4 P. M.; it gains $1\frac{1}{2}$ min. an hour; what is the true time when its hands are at right angles for the fourth time after 4?

10. A clock indicates correct time when its hands are together between 2 and 3 o'clock; if it had been losing 2 min. every hour, what time did it indicate at 12 noon?

11. A clock, in which the hour-hand has been displaced, shows the time to be 16 minutes past 3, and the two hands are together: the time is between 3 and 4 o'clock. Find by how many minute-divisions the hand has been displaced.

12. If the hands of a clock come together every 63 minutes (true time), how much does the clock gain or lose in a day?

PROBLEMS CONCERNING TIME AND DISTANCE.

202. *Example* 1. A passenger train leaves Calcutta at 4 P. M. and travels at the rate of 20 miles an hour; the mail train leaves Calcutta at 9 P. M. and travels, on a parallel line of rails, at the rate of 30 miles an hour: when and where will the second train overtake the first?

The first train has started 5 hours before the second; and is therefore (20×5) or 100 miles away when the second train starts. Therefore the second train has to gain 100 miles on the first, at the rate of 10 (*i.e.*, $30 - 20$) miles an hour.

Second train gains 10 miles in 1 hour on the first,

$\therefore$ .... 100 .... 10 hours................ ;

$\therefore$ the time required is 10 hours after the second train starts: and $\therefore$ the second overtakes the first (30×10) or 300 miles from Calcutta.

Example 2. A hare, pursued by a greyhound, is 30 yards before him at starting; whilst the hare takes 4 leaps the dog takes 3; in one leap the hare goes $1\frac{1}{2}$ yards, and the dog, $2\frac{1}{2}$ yards: how far will the hare have gone when she is caught by the hound?

Whilst the hare runs $(4\times 1\frac{1}{2})$ yd., or 6 yd., the dog runs $(3\times 2\frac{1}{2})$ yd., or $7\frac{1}{2}$ yd. Hence

The dog gains $1\frac{1}{2}$ yd. whilst the hare runs 6 yd.,

$\therefore$ .... 3 yd. .... 12 yd.,

$\therefore$ .... 30 yd. .... 120 yd.;

$\therefore$ the required distance is 120 yd.

Example 3. *A* starts from *P* to walk to *Q*, a distance of $51\frac{3}{4}$ miles, at the rate of $3\frac{3}{4}$ miles an hour; an hour later *B* starts from *Q* for *P* and walks at the rate of $4\frac{1}{4}$ miles an hour: when and where will *A* meet *B*?

A has already gone $3\frac{3}{4}$ miles when *B* starts. Of the remaining 48 miles, *A* walks $3\frac{3}{4}$ and *B* walks $4\frac{1}{4}$ in one hour; that is, they together pass over $(3\frac{3}{4}+4\frac{1}{4})$ or 8 miles in one hour. Therefore 48 miles are passed over in $\frac{48}{8}$ or 6 hours. Therefore *A* meets *B* in 6 hours after *B* started. And therefore they meet at a distance of $4\frac{1}{4}\times 6$ or $25\frac{1}{2}$ miles from *Q*.

Example 4. Two trains, 77 yd. and 99 yd. long respectively, run at the rates of 25 and 20 miles an hour respectively on parallel rails in opposite directions: how long do they take to pass each other? How long would they take to pass each other if they were running in the same direction? How long would a person sitting in the first train take to pass the other?

(i) The two trains running in opposite directions will pass each other in the time in which (77+99) or 176 yards are passed over at the rate of (25+20) or 45 miles an hour.

Now, 45 miles are passed over in 1 hour,

i.e., 45×1760 yd. .... 1 hour,

$\therefore$ 176 yd. .... $\frac{1}{450}$ hour;

$\therefore$ the time required $=\frac{1}{450}$ hr., or 8 seconds.

(ii) When the trains run in the same direction they pass each other in the time in which (77+99) or 176 yards are passed over at the rate of (25−20) or 5 miles an hour. The time required will be found to be 72 seconds.

(iii) First, when the trains are running in opposite directions, a person sitting in the first train will pass the other in the time in which 99 yd. (*i.e.*, the length of the second train) are passed over at the rate of (25+20) or 45 miles an hour. The required time will be found to be $4\frac{1}{2}$ seconds.

Secondly, when the trains run in the same direction, 99 yd. are to be passed over at the rate of (25−20) or 5 miles an hour. The required time is $40\frac{1}{2}$ seconds.

Example 5. A man rows down a river 18 miles in 4 hours with the stream, and returns in 12 hours: find the rate at which he rows, and the rate at which the stream flows.

He rows 18 miles in 4 hours down the stream; therefore he rows $\frac{18}{4}$ or $4\frac{1}{2}$ miles an hour down the stream.

Again, he rows 18 miles in 12 hours up the stream; therefore he rows $\frac{18}{12}$ or $1\frac{1}{2}$ miles an hour up the stream.

$\therefore$ $4\frac{1}{2}$ miles an hour is the sum of the rate at which the man rows and the rate at which the stream flows; and $1\frac{1}{2}$ miles an hour is their difference. Hence the rates are 3 miles and $1\frac{1}{2}$ miles an hour respectively.

Example 6. If a snail, on the average, creep 31 inches up a pole during 12 hours in the night, and slip down 16 inches during the 12 hours in the day, how many hours will he be in getting to the top of a pole 35 feet high?

Length of the pole $=420$ in. Now in 24 hours the snail creeps up $(31-16)$ in. or 15 in.; therefore in (24×26) hr. the snail creeps up (15×26) in., or 390 in.; therefore he has $(420-390)$ in. or 30 in. more to get up. And he goes over 31 in. in 12 hr., and therefore over 30 in. in $\frac{12\times 30}{31}$ hr. Therefore he reaches the top in $(24\times 26+\frac{12\times 30}{31})$ hr., or in $635\frac{19}{31}$ hours. [The number of days (26) has been so determined that (420 in. $-$ 15 in. $\times$ 26) may be equal to 31 in. or just less than 31 in.]

EXAMPLES. 132.

1. One man takes 100 steps a minute, each 2 ft. long; another walks 4 miles an hour; if they start together, how soon will one of them be 38 yards ahead of the other?

2. A person wishing to go from *A* to *B* walked for $4\frac{1}{3}$ hours at the rate of 1 mile in $21\frac{3}{8}$ min., he then rode for $16\frac{1}{4}$ hours three times as fast as he walked, and then had to travel by rail for $10\frac{5}{6}$ hours three times as fast as he rode; find the distance from *A* to *B*.

3. A train leaves Calcutta at 7-30 A. M. and travels 25 miles an hour; another train leaves Calcutta at noon and travels 40 miles an hour: when and where will the second train overtake the first?

4. A train going 30 miles an hour leaves Calcutta for Allahabad (600 miles) at 9 P. M., another train going 40 miles an hour leaves Allahabad for Calcutta at the same time; when and where will they pass each other?

5. Two trains, each 88 yards long, are running in opposite directions on parallel rails, the first at 40 miles an hour, the

other at 35 miles an hour; how long will they take to pass each other?

6. In the above example, if the trains run in the same direction, how long will a person sitting in the faster train take to pass the other?

7. A man rows down a river 15 miles in 3 hours with the stream and returns in $7\frac{1}{2}$ hours; find the rate at which he rows, and the rate at which the stream flows.

8. A man rows 12 miles in 5 hours against the stream, the rate of which is 4 miles an hour: how long will he be rowing 15 miles with the stream?

9. A policeman goes after a thief who has 100 yards' start; if the policeman run a mile in 6 minutes, and the thief a mile in 10 minutes, how far will the thief have gone before he is overtaken?

10. A man starts at 7 A. M. and travels at the rate of $4\frac{3}{4}$ miles an hour: at 8-15 A. M. a coach starts from the same place and follows the man, travelling at the rate of $6\frac{1}{2}$ miles an hour: at what o'clock will the coach overtake the man?

11. *A* starts from Allahabad to Cawnpore and walks at the rate of 5 miles an hour; *B* starts from Cawnpore 3 hours later and walks towards Allahabad at the rate of $4\frac{1}{2}$ miles an hour: if they meet in 11 hours after *B* started, find the distance from Allahabad to Cawnpore.

12. *A* starts from Calcutta to Hugli (24 miles) at 6 A. M. walking 4 miles an hour; *B* starts from Calcutta an hour later and reaches Hugli one hour before *A*; where did they meet?

13. A man walks to a town at the rate of $3\frac{1}{2}$ miles an hour and rides back at the rate of 6 miles an hour: how far has he walked, the whole time occupied having been 3 hours 10 minutes?

14. *A* and *B* run a mile in opposite directions: while *A* runs 6 yards *B* runs 5; *B* gets 9 seconds' start, during which time he runs $22\frac{1}{2}$ yards; find when he will pass *A*.

15. A train leaves Calcutta at 7 A. M. and reaches Burdwan at 11 A. M.; another train leaves Burdwan at 8 A. M. and reaches Calcutta at 10-30 A. M.: at what hour do they meet?

16. A train starts from *P* for *Q* travelling 20 miles an hour; $1\frac{1}{2}$ hours later another train starts from *P* and travelling at the rate of 30 miles an hour reaches *Q* $2\frac{1}{2}$ hours before the first train: find the distance from *P* to *Q*.

17. A horseman leaves Madras at 10 A. M. and in 5 hours overtakes a coach which left Madras at 9 A. M. If the coach had been 2 miles farther on the road when the horseman started, it would

have been overtaken in 7 hours. Find the rates of the horseman and the coach.

18. A and B start at the same time from Patna and Bankipore, and proceed towards each other at the rates of 3 and 4 miles per hour respectively. They meet when B has walked one mile farther than A. Find the distance between Patna and Bankipore.

18a. A, B and C start from the same place at intervals of an hour and walk at the rate of 3, 4 and 5 miles an hour respectively. A starts first, but when he is overtaken by B he returns towards the starting-place; find the distance from the starting-place where he would meet C.

19. A man rides at the rate of 11 miles an hour, but stops 5 minutes to change horses at the end of every 7th mile; how long will he take to go a distance of 94 miles?

20. A man rides at the rate of 10 miles an hour, but stops 10 minutes to change horses at the end of every 12th mile; how long will he take to go a distance of 96 miles?

21. If a gun fire 7 shots every 9 minutes, how many will it fire in an hour?

22. A monkey, climbing up a greased pole, ascends 10 ft. and slips down 3 ft. in alternate minutes. If the pole is 63 ft. high, how long will it take him to reach the top?

23. A vessel has 2 pipes attached to it, 1 to supply and 1 to draw off. The supply-pipe can fill the vessel in 40 minutes, and the waste-pipe can empty it in an hour. If the supply-pipe and waste-pipe are kept open in alternate minutes, in what time will the vessel be filled?

24. A boy and a girl began to fill a cistern: the boy brings a quart at the end of every 2 minutes and the girl brings a pint every 3 minutes. In what time will the cistern be filled, if it holds $4\frac{1}{2}$ gallons?

203. *Example.* A, B and C start from the same point and travel round an island 30 miles in circumference, A and B travelling in the same direction and C in the opposite direction. If A travels at the rate of 5, B at the rate of 7 and C at the rate of 8 miles an hour, in how many hours will they all come together again?

B gains 2 miles on A in 1 hour; $\therefore$ he gains 30 miles or a complete circuit in $\frac{30}{2}$ hr., that is, A and B are together at the end of every 15 hours. A and C together pass over 13 miles in 1 hr.; $\therefore$ they come together every $\frac{30}{13}$ hours. And therefore A, B and C will come together at the end of any number of hours which is a common multiple of 15 and $\frac{30}{13}$; but the L. C. M. of 15 and $\frac{30}{13}$ is 30; therefore A, B, C are first together at the end of 30 hours.

EXAMPLES. 133.

1. A and B start together from the same point to walk round a circular course, 10 miles long; A walks 4 miles and B 3 miles an hour. When will they next meet, (i) if they walk in the same direction, (ii) if they walk in opposite directions?

2. A takes 3 hours and B takes 5 to walk round a park. If they start together, when will they next meet, supposing (i) that they walk in the same direction, (ii) that they walk in opposite directions?

3. A, B, C start from the same point and travel in the same direction round an island 63 miles in circumference. A at the rate of 10, B at the rate of 12, and C at the rate of 16 miles a day: in how many days will they come together again?

4. A can go round an island in 15 days, B can go round it in 20 days and C in 25 days. If they start simultaneously from the same point, A and B travelling in one direction and C in the opposite direction, in how many days will they come together again? In how many days will they come together again at the starting point?

5. Three boys agree to start together from the same point and run round a circular park 6 miles in circumference; they run at the rates of 3, 5 and 7 miles per hour respectively; in how many hours will they come together again? In what time will they come together again at the point from which they started?

RACES AND GAMES OF SKILL.

204. *Example* 1. A can beat B by 40 yards in a mile race: B can beat C by 20 yards in a mile race: if A and C run a mile, by how much will A win?

A can run 1760 yards while B runs 1720,

$\therefore$ A .... $\frac{1760}{43}$ .... B 40,

$\therefore$ A .... $\frac{1760\times 44}{43}$ .... B 1760,

[but B .... 1760 .... C ...... 1740,]

$\therefore$ A .... $\frac{1760\times 44}{43}$ .... C 1740,

$\therefore$ A .... 1760 .... C ...... $\frac{1740\times 43}{44}$ or $1700\frac{5}{11}$ yd.

$\therefore$ A will win by $(1760-1700\frac{5}{11})$ or $59\frac{6}{11}$ yards.

Example 2. A can give B 20 yards, and C 30 yards in a race of 200 yards: how many yards can B give C in 300 yards?

[Note.—"A can give B 20 yards in 200 yards" means that in a race of 200 yards A can give B 20 yards' start. Consequently while A runs 200 yards B runs 180 yards.]

While A runs 200 yards B runs 180,
and .... A ...... 200 .... C 170,
$\therefore$ .... B ...... 180 .... C 170,
$\therefore$ .... B ...... 60 .... C ...... $\frac{170}{3}$,
$\therefore$ .... B ...... 300 .... C ...... $\frac{170 \times 5}{3}$ or $283\frac{1}{3}$ yards.
$\therefore$ B can give C $(300 - 283\frac{1}{3})$ or $16\frac{2}{3}$ yards in 300.

Example 3. In a game of skill A can give B, and B can give C, 10 points out of a game of 50; how many should A give C?

[Note.—"A can give B 10 points out of a game of 50" means that while A makes 50 points B can make $(50 - 10)$ or 40 points.]

C can make 40 points while B makes 50,
$\therefore$ C .... 4 .... B 5,
$\therefore$ C .... 32 .... B 40;
but A .... 50 .... B 40;
$\therefore$ C .... 32 .... A .... 50.
$\therefore$ A can give C $(50 - 32)$ or 18 points in 50.

EXAMPLES. 134.

1. In a mile race A gives B 60 yards' start, and beats him by 28 yards. If A runs the mile in 5 minutes, how long will B take?

2. In a mile race A can beat B by 40 yards, and B can beat C by 40 yards: how many yards' start can A give C that there may be a dead heat?

3. A can give B 60 yards, and C 80 yards in a race of 500 yards: by how much could B beat C in a mile race?

4. A runs 15 yards while B runs 12; B runs 10 miles while C runs 12: if C runs a mile in 10 minutes, what time will A take to do it?

5. At a game of skill A can give B 15 points out of 50, and A can give C 10 points out of 40: which is the better player, B or C, and how many points can he give the other in 75?

6. A and B run a mile race; A runs the whole course at the rate of 100 yards per minute; B running at the rate of 80 yards per minute for 5 minutes, quickens his speed to 120 yards per minute: which wins? by how much? and by what time?

7. In a game of billiards A can give B 10 points, and C 14 points in 50 : how many can B give C so as to make an even match ?

8. A can give B 300 yards in 1 mile, and C can give B 700 yards in 2 miles ; if A and C run a mile, which will win and by how much ?

9. A can give B 100 yards' and C 150 yards' start in a mile ; B can give C a start of 5 seconds in a mile ; how long does each take to run half a mile ?

10. In a mile race A gives B 50 yards' start, and beats him by 38 yards ; B giving C 40 yards' start is beaten by 60 yards : if A and C run over the same course, which will win and by how much ?

11. At a game of rackets A can give B 8 points in 40, and B can give C 10 points in 50 : how many points could A give C in 25 ?

12. A can give B 20 yards' and C 30 yards' start, while B can give C 2 seconds' start in a race of 250 yards ; how long does each take to run 100 yards ?

13. One boy runs 200 yards and another 180 yards in a minute. How many yards' start must the second have that they may run a dead heat in a mile race ?

14. In a game at fives A can give B 3 points out of 15, and A can give C 7 points : how many points can B give C so as to make an even match ?

15. A and B run a mile and A wins by half a minute. A and C run a mile and A beats C by 88 yards. B and C run and B wins by 20 seconds. In what time can each run a mile ?

16. A beats B by 20 yards, C beats D by 60 yards, and B beats D by 40 yards, in a mile race. If A and C run, which will win and by how much ?

CHAIN RULE.

205. *Example* 1. If 8 rupees are worth 15 shillings, and 25 shillings are worth 6 dollars, how many dollars are equal to 45 rupees ?

$$\text{R}8 = 15s., \qquad \therefore \ \text{R}1 = \tfrac{15}{8}s.$$
$$25s. = 6 \text{ dollars}, \qquad \therefore \ 1s. = \tfrac{6}{25} \text{ dollars}.$$
$$\therefore \ \text{R}45 = 45 \times \tfrac{15}{8}s.$$
$$= 45 \times \tfrac{15}{8} \times \tfrac{6}{25} \text{ dollars, or } 20\tfrac{1}{4} \text{ dollars}.$$

Example 2. If A in 3 days can do as much work as B in 4 days, and B in 5 days can do as much as C in 6 days, how long will A require to do a piece of work which C can do in 16 days?

What C can do in 6 da. B can do in 5 da.,

$\therefore$ .... C .... 1 .... B .... $\frac{5}{6}$ ....,

and .... B .... 4 .... A .... 3 ....,

$\therefore$ .... B .... 1 .... A .... $\frac{3}{4}$ ....;

$\therefore$ What C can do in 16 days B can do in $16\times\frac{5}{6}$ days,

$\therefore$ .... C .... A .... $16\times\frac{5}{6}\times\frac{3}{4}$ days

or 10 days.

EXAMPLES. 135.

1. If 25 rupees are worth 46 shillings, 20 shillings are worth 25 francs, and 240 francs are worth 47 dollars, how many dollars are equivalent to 40 rupees?

2. If ₹8=15*s*., £3=20 thalers, and 25 thalers=93 francs, express a franc in Indian money.

3. If 72 carlini=25 shillings, 4 shillings=5 francs, and 8 scudi=45 francs, how many scudi are equal to 1295 carlini?

4. If 5 chickens cost as much as 4 ducks, 6 ducks cost as much as 3 geese, and 7 geese cost as much as 5 turkeys, what is the price of a chicken when a turkey costs ₹8?

5. If 5 lb. of tea be worth 3 lb. of coffee, 5 lb. of coffee be worth 2 lb. of sugar, and 7 lb. of sugar be worth 30 lb. of rice, how many pounds of tea must be given in exchange for 20 lb. of rice?

6. If 12 oxen eat as much as 29 sheep, 15 sheep eat as much as 25 hogs, 17 hogs eat as much as 3 camels, and 8 camels eat as much as 13 horses, how many horses will eat as much as 1632 oxen?

7. If A can do as much work in 4 days as B can do in 5, and B can do as much in 6 days as C in 7; in what time will C do a piece of work which A can do in a week?

8. If A can do as much work in $1\frac{1}{2}$ days as B can do in 2, and B can do as much in $2\frac{1}{2}$ days as C in 3; in what time will A and B together do a piece of work which C can do in 10 days?

9. While A does $\frac{1}{3}$ of a piece of work B does $\frac{1}{4}$, and while B does $\frac{1}{5}$ C does $\frac{1}{6}$; in how many hours will C finish a piece of work which A finishes in 20 hours?

10. If 3 ducks are worth 4 chickens, and 3 geese are worth 10 ducks, find the value of a goose, a pair of chickens being worth 4*a*. 6*p*.

XXXVI. COMPLEX PROBLEMS.

206. In the problems in the preceding section we have found the change in one quantity corresponding to the change in *one* other. In the following examples we shall have to find the change in one quantity corresponding to the changes in *two* others.

Example 1. If 15 horses can plough 12 acres in 10 days, in how many days can 9 horses plough 18 acres?

15 horses can plough 12 acres in 10 days,

∴ 1 horse .... 12 acres in (10 × 15) days,

∴ 1 horse .... 1 acre in $\frac{10\times15}{12}$ days,

∴ 9 horses .... 1 acre in $\frac{10\times15}{12\times9}$ days,

∴ 9 horses .... 18 acres in $\frac{10\times15\times18}{12\times9}$ days,

or 25 days. *Ans.*

Note. We might use 3 horses and 6 acres as common units with advantage. Thus:

15 horses can plough 12 acres in 10 days,

∴ 3 horses .... 12 acres in 10 × 5 days,

∴ 3 horses .... 6 acres in $\frac{10\times5}{2}$ days,

∴ 9 horses .... 6 acres in $\frac{10\times5}{2\times3}$ days,

∴ 9 horses .... 18 acres in $\frac{10\times5\times3}{2\times3}$ days,

or 25 days. *Ans.*

Example 2. If 6 men earn R15 in 10 days, how much do 8 men earn in 7 days?

In 10 days 6 men earn R15,

∴ in 1 day 6 men earn R$\frac{15}{10}$ or R$\frac{3}{2}$,

∴ in 1 day 1 man earns R$\frac{3}{2\times6}$ or R$\frac{1}{4}$,

∴ in 7 days 1 man earns R$\frac{7}{4}$,

∴ in 7 days 8 men earn R$\frac{7\times8}{4}$ or R14. *Ans.*

Example 3. If 6 men can do a piece of work in 8 days, how many men can do a piece of work 4 times as great in $\frac{1}{3}$ of the time?

The work can be done in 8 days by 6 men,

∴ .... $\frac{8}{3}$ .... 18 men,

∴ 4 times the work .... $\frac{8}{3}$ .... 72 men. *Ans.*

Example 4. If the sixpenny loaf weigh 8 oz. when wheat is 15*s.* a bushel, what ought a bushel of wheat to be when the fourpenny loaf weighs 12 oz.?

	Sixpenny	loaf weighs	8 oz. when wheat is	15*s*. a bushel,
∴	penny	loaf weighs	8 oz.	$\frac{5}{2}$*s*. ....,
∴	penny	loaf weighs	1 oz.	20*s*. ....,
∴	fourpenny	loaf weighs	1 oz.	80*s*. ....,
∴	fourpenny	loaf weighs	12 oz.	$\frac{20}{3}$*s*. ....,
				or 6*s*. 8*d*. a bushel.

***Example* 5.** If 5 cannon, which fire 3 rounds in 5 minutes, kill 135 men in $1\frac{1}{2}$ hours, how many cannon, which fire 5 rounds in 6 minutes, will kill 250 men in 1 hour?

	In	54 rounds	135 men are killed by	5 cannon,
∴	...	1 round	135	5×54 ....,
∴	...	1 round	1 man is	$\frac{5\times 54}{135}$ ....,
∴	...	50 rounds	1	$\frac{5\times 54}{135\times 50}$ ....,
∴	...	50 rounds	250 men are	$\frac{5\times 54\times 250}{135\times 50}$..........,
				or 10 cannon.

EXAMPLES. 136.

1. If 5 men earn £3 in 12 days, in how many days will 8 men earn £4?

2. If 10 horses can plough 50 acres in 20 days, how many acres will 12 horses plough in 15 days?

3 If 24 horses eat 9 bushels of corn in 21 days, for how many days will 33 bushels feed 7 horses?

4. If 30 men can build a wall 20 ft. high in 15 days, how many men will it take to build one 25 ft. high in $7\frac{1}{2}$ days?

5. If 12 horses are fed for 17 days at a cost of ₨110. 8*a*., how many horses can be fed for 27 days at a cost of ₨117?

6. If 10 fires consume 75 maunds of coal in 14 days, in how many days will 18 fires consume 100 maunds?

7. If the carriage of 10 md. 20 seers for 250 miles be ₨41. 0*a*. 3*p*., what should be paid for the carriage of 12 md. for 200 miles?

8. If the wages of 13 men for 25 days amount to ₨20. 5*a*., how many men must work for 16 days to receive ₨30?

9. What is a month's rent for $116\frac{2}{3}$ bighas of land, if ₨22. 8*a*. per annum be given for 9 bighas?

10. If 14 persons can live on ₨1400 for 28 months, how long can 18 persons live on ₨1350?

11. If 5 men dig a trench $7\frac{1}{2}$ yd. long in 21 days, how many men can dig a similar trench 20 yd. long in 35 days?

12. If 20 pumps can raise 1250 maunds of water in 5 hours, how many pumps can raise 750 maunds of water in 10 hours?

13. If 20 men do a piece of work in 13 days, in what time can 15 men do another piece of work $2\frac{1}{2}$ times as great?

14. If 10 men do a piece of work in 8 days, how many men will do a piece of work, 4 times as great, in $\frac{1}{3}$ of the time?

15. If the fourpenny loaf weighs 10 oz. when wheat is 50*s.* a quarter, what should a threepenny loaf weigh when wheat is 55*s.* a quarter?

16. If 3 lb. loaf cost 8*d.* when corn is 30*s.* per bushel, how much ought the 5 lb. loaf to cost when corn is 36*s.* per bushel?

17. If I get 1 lb. weight of bread for $7\frac{1}{2}d.$ when wheat is 15*s.* a bushel, what ought a bushel of wheat to be when I get 12 oz. of bread for 4*d.*?

18. If 14 men in 20 days of $12\frac{1}{2}$ hours each earn R456. 4*a.*, how many hours a day should 24 men work to earn R547. 8*a.* in 21 days, at the same rate?

19. If 15 men can do a piece of work in 12 days of 6 hours each, how many men will it take to do 5 times the amount if they work 20 days of 10 hours each?

20. If a man complete a journey of 1980 miles in 18 days, travelling 11 hours a day, in how many days would he travel 540 miles, going 6 hours a day at the same rate?

21. When rice is R2. 8*a.* a maund, 10 men can be fed for 12 days at a certain cost; how many men can be fed for 4 days at the same cost, when rice is R3 a maund?

22. When flour is R4 a maund, 16 men can be fed for 5 days at a cost of R8; for how many days can 12 men be fed at a cost of R10. 8*a.*, when flour is R3. 8*a.* per maund?

23. If 15 men can build a wall 270 ft. long, 5 high and 2 thick, in 18 days, in how many days will 16 men build a wall 180 ft. long, 4 high and 3 thick?

24. If 10 men working 6 hours a day dig a trench 105 ft. long, 4 wide and 2 deep in 6 days, how many hours a day must 264 men work in order to dig a trench 126 ft. long, 20 wide and 11 deep in 10 days?

25. A garrison of 1200 men is provisioned for 50 days, allowing 10 oz. per man per day; if it is re-inforced by 300 men, to what must the daily allowance be reduced that the provisions may last the increased number of men 60 days?

26. If the carriage of goods weighing 2 cwt. 3 qr. 6 lb. for 300 miles cost £6 . 10 . 10, what will be the charge for carrying

2 wagon-loads of the same, each weighing 14 cwt. 0 qr. 4 lb., 450 miles?

27. If the gas for 6 burners, 6 hours every day, for 8 days cost ₨4. 8*a.*, how many burners may be lighted 5 hours every evening for 10 days at the cost of ₨6. 4*a.*?

28. If 3 cannon, firing 4 rounds in 6 minutes, kill 250 men in half an hour, how many cannon, firing 3 rounds in 5 minutes, will kill 600 men in an hour?

29. If 15 men can make an embankment, 966 yd. long, in 8 days, working $10\frac{1}{2}$ hours daily, how many men would be required to make an embankment, 575 yd. long, in 12 days, working $7\frac{1}{2}$ hours daily, 8 extra men being taken on during the last 2 days?

30. If 50 men, working 8 hours a day, dig in 5 days, a trench of 275 cu. yd.; in how many days of 10 hours each could 40 men dig a trench of 330 cu. yd., when the hardness of the ground in the first case is twice that in the second, and 3 men of the former company can do the work of 4 men of the latter?

31. If 6 men, working 8 hours a day, can mow 60 acres in 4 days; in how many days will 4 men, two of whom work 10 hours and two 7 hours a day, mow 85 acres?

32. If 6 men and 8 boys can reap a field of 15 acres in 4 days, how many acres will 7 men and 4 boys reap in 9 days, two boys reaping as much as a man in the same time?

33. If 4 horses eat as much as 18 sheep, and if 5 horses and 30 sheep can be kept for 15 days at a cost of ₨51 . 3 . 6, at what cost can 7 horses and 15 sheep be kept for 20 days?

34. The rent of a farm of $41\frac{1}{4}$ acres for 39 months was ₨89. 6*a.*; what would be the area of another farm, the rent of which for 33 months was ₨103. 2*a.*, 4 acres of the latter being worth as much as 3 acres of the former?

35. A vessel with a crew of 27 men, provisioned for 90 days at the rate of 22 oz. a day per man, was, after 27 days, forced by stress of weather to lie at anchor for a fortnight, at the end of which time 3 men died; how must the provisions be apportioned that they may hold out the extra time?

36. If 10 men or 16 boys, working 6 hours a day, can do a piece of work in 20 days, how many hours a day must 7 men and 8 boys work to do another piece of work 3 times as great in 15 days?

37. If 5 men, 8 women or 12 boys can do a piece of work in 16 days, working 7 hours a day, how many men, with the assistance of 4 women and 6 boys, will be able to do another piece of work $2\frac{1}{2}$ times as great in 35 days, working 5 hours a day?

207. The following problems are of a different class.

Example 1. The price of 5 horses and 6 oxen is ₨680, that of 4 horses and 7 oxen is ₨610; find the price of an ox.

The price of 5 horses and 6 oxen = ₨680,

∴ .... 20 .... 24 .... = ₨2720. .... (i)

Again .... 4 .... 7 .... = ₨610,

∴ .... 20 .... 35 .... = ₨3050; ....(ii)

∴ the price of 11 oxen = ₨3050 − ₨2720 [subtracting (i) from (ii)]

= ₨330;

∴ the price of 1 ox = ₨30.

Example 2. 3 men and 5 boys can do $\frac{19}{20}$ of a piece of work in 3 days; 4 men and 8 boys can do $\frac{14}{15}$ of it in 2 days: in what time can a boy do the whole work?

In 3 days 3 men and 5 boys can do $\frac{19}{20}$,

∴ ... 1 day 3 .... 5 .... $\frac{19}{60}$,

∴ ... 1 day 12 .... 20 .... $\frac{19}{15}$. .... (i)

Again ... 2 days 4 .... 8 .... $\frac{14}{15}$,

∴ ... 1 day 4 .... 8 .... $\frac{7}{15}$,

∴ ... 1 day 12 .... 24 .... $\frac{7}{5}$; ....(ii)

∴ in 1 day 4 boys can do $(\frac{7}{5}-\frac{19}{15})$ of the work,

[subtracting (i) from (ii)]

i.e., ... 4 boys can do $\frac{2}{15}$ of the work,

∴ .... 1 boy can do $\frac{1}{30}$ of the work,

∴ 1 boy can do the whole work in 30 days.

EXAMPLES. 137.

1. If 9 horses and 7 cows cost ₨770, and 5 horses and 9 cows cost ₨530; find the price of a cow.

2. The price of 5 maunds of flour and 6 maunds of rice is ₨39, and that of 7 maunds of flour and 4 maunds of rice is ₨37; find the price of one maund of flour and of one maund of rice.

3. If 10 rupees and 11 shillings weigh 2760 grains, and 8 rupees and 10 shillings weigh 2312$\frac{8}{11}$ grains, find the weight of a rupee and of a shilling.

4. If 7 sheep and 9 pigs cost ₨107, and 9 sheep and 7 pigs cost ₨101, how much will 1 sheep and 1 pig cost?

5. The cost of 4 chairs and 5 tables is ₨120, and that of 5 chairs and 4 tables ₨105; find the price of a chair and of a table.

6. 2 men and 3 boys can do $\frac{3}{4}$ of a piece of work in 6 days; 3 men and 5 boys can do $\frac{19}{24}$ of it in 4 days. In what time can a boy do the whole work?

7. 7 men and 8 boys can do a piece of work in 2 days; 4 men and 12 boys can do $\frac{29}{56}$ of the work in 1 day. In what time can a man do the work?

8. 5 men and 6 boys can do $\frac{3}{5}$ of a piece of work in 3 days; 10 men and 18 boys can do the whole work in 2 days. In what time will a man and a boy be able to do double the work?

9. If 6 men and 2 boys can reap 13 acres in 2 days, and 7 men and 5 boys can reap 33 acres in 4 days, how long will it take 2 men and 2 boys to reap 10 acres?

10. If 2 boys and 1 man can do a piece of work in 4 hours, and 2 men and 1 boy can do the same in 3 hours, find in what times a man, a boy, and a man and a boy together, respectively, could do the same.

11. On a piece of work 4 men and 5 boys are employed, who do $\frac{1}{2}$ of it in 6 days; after this, 1 man and 2 boys more are put on, and $\frac{1}{3}$ more is done in 3 days; how many more men must be put on to finish the work in one more day?

12. A cistern containing 210 buckets may be filled by two pipes. When the first pipe has been open 4 and the second 5 hours, 90 buckets of water were obtained. When the 1st was open 7 and the 2nd $3\frac{1}{2}$ hours, 126 buckets were obtained. In what time will the cistern be full, if both pipes work?

XXXVII. RATIO AND PROPORTION.

208. The **ratio** of one quantity to another of the same kind is that which expresses the relative greatness of the first quantity with respect to the second.

Hence, the ratio of one quantity to another (of the same kind) is determined by the *fraction* whose numerator is the measure of the first quantity and whose denominator is the measure of the second quantity, both the quantities being expressed in terms of the same unit.

Thus, the ratio of 3*s*. to 5*s*. is determined by the fraction $\frac{3}{5}$; of 2 yd. to 5 ft. by the fraction $\frac{6}{5}$.

The first of the two quantities forming a ratio is called the **antecedent** and the second is called the **consequent** of the ratio; the two together are called the **terms** of the ratio. The ratio of 3*s*. to 5*s*. is written 3*s*. : 5*s*.

Note. The *inverse* ratio of 3*s*. to 5*s*. is the ratio of 5*s*. to 3*s*.

209. The value of a ratio does not depend upon the nature of the quantities involved. Thus, the ratios, 2 yd. : 5 yd., 2*s.* : 5*s.*, 2 lb. : 5 lb., are all equal, each of these being determined by the fraction $\frac{2}{5}$. Hence, in investigating the properties of ratios, we usually consider the terms to be numbers, because numbers measure quantities of all kinds.

210. The value of a ratio is not altered by multiplying or dividing both its terms by the same number. Thus, the ratios, 2 : 3, 4 : 6, 80 : 120, are all equal.

211. Ratios are **compounded** by taking the product of the antecedents for a new antecedent and the product of the consequents for a new consequent. Thus the ratio compounded of the ratios, 2 : 3 and 6 : 7 is $2\times6 : 3\times7$ or 4 : 7.

212. Four quantities are said to be **in proportion** or **proportionals** when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus 3, 4, 9, 12 are in proportion ; since the ratio of 3 to 4 is equal to the ratio of 9 to 12.

N. B. When four quantities are in proportion, it is not necessary that all of them should be of the same kind ; it is only necessary that the first two should be of the same kind, as also the second two.

The existence of proportion among the numbers is denoted thus :—

$$3 : 4 = 9 : 12,$$

which is read "3 to 4 equals 9 to 12" ;

or thus :— $$3 : 4 :: 9 : 12,$$

which is read "3 is to 4 as 9 is to 12".

Of this proportion 3 and 12 are called the **extremes**, and 4 and 9, the **means** ; 12 is called a **fourth proportional** to 3, 4 and 9.

213. When four quantities are in proportion so that

first : second :: third : fourth ;

then also, second : first :: fourth : third ;

and fourth : third :: second : first.

Also, if the quantities are all of the same kind,

first : third :: second : fourth.

214. When four numbers are in proportion, the product of the extremes is equal to the product of the means.

For example, $3 : 4 = 6 : 8$, and we have $3 \times 8 = 4 \times 6$.

Hence also, an extreme=product of the means ÷ the other extreme; and, a mean=product of the extremes ÷ the other mean.

215. Three quantities of the same kind are said to be **in continued proportion** when the ratio of the first to the second is equal to the ratio of the second to the third. The second quantity is called **a mean proportional** between the first and third; and the third quantity is called **a third proportional to** the first and second.

Thus, 2, 4 and 8 are in continued proportion; for $2 : 4 = 4 : 8$; 4 is a mean proportional between 2 and 8; and 8 is a third proportional to 2 and 4.

It is obvious that the square of the mean proportional between two numbers is equal to their product.

216. *Example* 1. Find a fourth proportional to 3, 9 and 4.

$$3 : 9 = 4 : \text{number required},$$

$$\therefore \text{number required} = \tfrac{9 \times 4}{3} = 12.$$

Example 2. Find the number which has the same ratio to 20 that 3 has to 5.

$$3 : 5 = \text{number required} : 20,$$

$$\therefore \text{number required} = \tfrac{3 \times 20}{5} = 12.$$

Example 3. Find a mean proportional between 3 and 12.

Square of the number required $= 3 \times 12 = 36$;

$\therefore$ the number required $= \sqrt{36} = 6$.

Example 4. A, B, C, D are quantities of the same kind; and the ratio of A to B is $3 : 4$, of B to C is $5 : 7$, and of C to D is $8 : 9$. Find the ratio of A to D.

$$\text{Now,} \quad \frac{A}{B} = \frac{3}{4}, \quad \frac{B}{C} = \frac{5}{7} \text{ and } \frac{C}{D} = \frac{8}{9};$$

$$\therefore \quad \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{3}{4} \times \frac{5}{7} \times \frac{8}{9}, \text{ or } \frac{A}{D} = \frac{10}{21};$$

that is, $A : D :: 10 : 21$.

Note. We find the **continued ratio** of A, B, C and D, that is, we compare A, B, C and D, thus :

$$\left.\begin{aligned} A : B &= 3 : 4, \\ B : C = 5 : 7 = 1 : \tfrac{7}{5} &= 4 : \tfrac{28}{5}, \\ C : D = 8 : 9 = 1 : \tfrac{9}{8} &= \tfrac{28}{5} : \tfrac{63}{10}, \end{aligned}\right\}$$

We change the terms of the ratios in such a way that each antecedent may be equal to the preceding consequent.

$$\therefore \quad A : B : C : D = 3 : 4 : \tfrac{28}{5} : \tfrac{63}{10}$$
$$= 30 : 40 : 56 : 63 ;$$

which is read "A is to B is to C is to D as 30 is to 40 is to 56 is to 63."

And A, B, C, D are said to be **in proportion of** or **proportional to** 30, 40, 56, 63.

Example 5. A mixture (42 gallons) contains wine and water in the ratio of 5 to 2 : find the quantities of wine and water in the mixture.

If the mixture be divided into 7 (*i.e.*, $5+2$) equal parts, 5 of the parts will be wine and 2 water.

$\therefore$ The quantity of wine $= \frac{42}{7} \times 5$ gallons $= 30$ gallons ;

and the quantity of water $= \frac{42}{7} \times 2$ gallons $= 12$ gallons.

Example 6. A mixture (40 gallons) contains wine and water in the ratio of 3 to 1 ; how much water must be added to it that the ratio of wine to water may be $5 : 2$?

We find, as in the preceding example, that the mixture contains 30 gall. wine and 10 gall. water. Now while the wine remains the same 30 gallons, the water is to be increased so that the ratio of wine to water may be $5 : 2$; but $5 : 2 = 30 : 12$; $\therefore$ $(12-10)$ gall. or 2 gall. of water must be added.

EXAMPLES. 138.

Find the value of each of the following ratios in its simplest form :

1. $15 : 21$.
2. Rs.39 : Rs.65.
3. £3 : £5. 10*s*.
4. 360 in. : 270 in.
5. 350 lb. : 725 lb.
6. $2^\circ . 5' : 3^\circ$.
7. $3\frac{3}{4} : 5\frac{5}{8}$.
8. $2\frac{3}{6} : 4\cdot\dot{3}$.
9. 3 yd. : 7 ft. 6 in.

Express in its simplest form the ratio compounded of the ratios,

10. $7 : 9$ and $45 : 28$.
11. $1 : 2$, $2 : 3$ and $3 : 4$.
12. $2\frac{1}{2} : 3\frac{1}{3}$ and $\cdot\dot{3} : \cdot 25$.
13. $4 : 7$, $5 : 8$ and $21 : 30$.

Compare the ratios,

14. $3 : 5$ and $7 : 8$.
15. $13 : 21$ and $18 : 29$.
16. $2 : 3$, $3 : 4$ and $4 : 5$.
17. $3 : 7$, $5 : 9$ and $7 : 11$.

Are the following in proportion?

18. 6, 11, 18, 33. 19. 5, 7, 20, 27. 20. ₨3, ₨2. 4*a*., 4, 3.

Find a fourth proportional to

21. 7, 9 and 8. 22. $2\frac{1}{2}$, 3 and $4\frac{1}{3}$. 23. ·2, ·02 and ·002.

24. ₨380, ₨570 and 12 lb. 25. 4 yd., 2 yd. 2 ft. and £2.

26. 12 acres, 27 ac. and 20 men. 27. 12 men, 9 men and £3.

28. 6 miles, 20 mi. and 9 hours. 29. 3 cwt., 84 lb. and £1. 8*s*.

Find a mean proportional between

30. 7 and 28. 31. 13 and 117. 32. 9464 and 5600.

33. $\frac{5}{28}$ and $\frac{20}{63}$. 34. $2\frac{1}{2}$ and $5\frac{5}{8}$. 35. ·3 and ·012.

Find a third proportional to

36. $2\frac{1}{4}$ and $7\frac{1}{2}$. 37. 7 and $5\frac{3}{8}$. 38. ₨2 and ₨1. 4*a*.

39. Compare the rates of two trains, one of which runs 17 miles in 2 hours and the other $12\frac{1}{2}$ miles in $2\frac{1}{2}$ hours.

40. $A : B = 3 : 4$, $B : C = \frac{3}{4} : \frac{4}{5}$; find the ratio of A to C.

41. If $A = \frac{4}{5}$ of B, and $B = 2\frac{1}{2}$ of C, find the ratio of A to C.

42. If, when A earns ₨4, B earns ₨5; and when B earns ₨6, C earns ₨7; and when C earns ₨8, D earns ₨9; compare the earnings of A, B, C and D.

43. Two sums of money are proportional to 7 and 8; the first is £2; what is the other?

44. The weights of equal volumes of gold and water are as 37 is to 2. If a cu. ft. of water weigh 1000 oz. find the weight of a cu. ft. of gold.

45. The ratio of the circumference of a circle to its diameter is 22 : 7; find the circumference of a circle 10 ft. 6 in. in diameter.

46. One man adds 5 seers of water to 15 seers of milk, and another 3 seers of water to 12 seers of milk; compare the amount of milk in the two mixtures.

47. While A makes a profit of £3, B makes £4; and while B makes a profit of £5, C makes £6; if A makes a profit of £20, how much does C make in the same time?

48. A mixture (50 gall.) contains wine and water in the ratio of 3 : 2; find the quantities of wine and water in the mixture.

49. A mixture (30 gall.) contains wine and water in the ratio of 7 to 3; how much water must be added to it that the ratio of wine to water may be 3 : 7?

50. A greyhound pursues a hare and takes 4 leaps for every 5 leaps of the hare, but 3 leaps of the hound are equal to 4 of the hare; compare the rates of hound and hare.

XXXVIII. RULE OF THREE.

217. Problems which we have solved by the Unitary Method may also be solved by the method of finding a fourth proportional to three given quantities.

Example 1. Find the price of 12 maunds of sugar, when the price of 5 maunds is R60.

Here we observe that if the weight be *increased* 2, 3...times, the price will also be *increased* 2, 3...times ; therefore the ratio of the two weights is equal to the ratio of the two corresponding prices.

Hence 5 md. : 12 md. :: R60 : the answer ;

$\therefore$ the answer $= \text{R}\frac{12 \times 60}{5} = \text{R}144$.

Example 2. If 12 men can do a piece of work in 5 days, in how many days will 15 men do it ?

Here we observe that if the number of men be *increased* 2, 3... times, the number of days will be *decreased* 2, 3...times ; therefore the *inverse ratio* of the numbers of men is equal to the *ratio* of the corresponding numbers of days.

Hence 15 men : 12 men :: 5 days : the answer ;

$\therefore$ the answer $= \frac{12 \times 5}{15}$ days $= 4$ days.

218. The above method of solving a problem by finding a fourth proportional to three given quantities is commonly known by the name of **Rule of Three.**

In the first problem we have an example of what is called the Rule of Three **Direct**, because there the *direct* ratio of the two weights is equal to the ratio of the corresponding prices.

In the second problem we have an example of what is called the Rule of Three **Inverse**, because there the *Inverse* ratio of the numbers of men is equal to the ratio of the corresponding numbers of days.

219. It is obvious that the second term in a proportion is greater or less than the first according as the fourth is greater or less than the third. Hence we may lay down the following general rule for arranging the terms in a Rule of Three question.

Denote the answer by the letter x and place it for the 4th term ; and of the three given quantities place that which is of the same kind as the answer, for the 3rd term. Next from the nature of the question determine whether the answer will be greater or less than the third term, and place the greater or less of the two remaining quantities for the 2nd term according as the answer is greater or less than the 3rd term ; then place the remaining quantity for the first term.

Note. In working, the two first quantities in the proportion must be replaced by the numbers which measure them in terms of the same unit.

Example 1. If the third class railway fare for 110 miles is ₨1 . 11 . 6, what is the fare for 350 miles ?

$$\begin{array}{l} \text{mi.} \quad \text{mi.} \quad ₨. \quad a. \quad p. \\ 110 : 350 :: 1 . 11 . 6 : x, \end{array}$$

i. e., $11 : 35 :: 1 . 11 . 6 : x$;

$$\therefore \quad x = \frac{₨1 . 11 . 6 \times 35}{11} = \frac{₨60 . 2 . 6}{11}$$

$$= ₨5 . 7 . 6. \quad \textit{Ans.}$$

Or thus : $\because$ ₨1 . 11 . 6 = 330*p.*

$$x = \tfrac{35 \times 330}{11} p. = 1050 p.$$

$$= ₨5 . 7 . 6$$

The latter method is the one more generally adopted. The learner should observe that the 3rd term being expressed in pies the answer obtained at the first instance is also in pies.

Example 2. If a quantity of rice serve 100 men for 15 weeks, how many men will it serve 6 weeks?

$$\begin{array}{ccccccc} \text{weeks} & & \text{weeks} & & \text{men} & & \\ 6 & : & 15 & :: & 100 & : & x, \end{array}$$

i.e., $2 : 5 :: 100 : x$,

$$\therefore \quad x = \tfrac{5 \times 100}{2} \text{ men} = 250 \text{ men.} \quad \textit{Ans.}$$

Example 3. A bankrupt's debts amount to £1320, and his assets (*i.e.*, the value of his property) are £990, how much can he pay in the pound ?

$$\begin{array}{l} £. \quad £. \quad £. \\ 1320 : 1 :: 990 : x, \end{array}$$

$$\therefore \quad x = £\tfrac{1 \times 990}{1320} = £\tfrac{3}{4} = 15s. \quad \textit{Ans.}$$

Example 4. A man, after paying an income-tax of 4*p.* in the rupee, has ₨4794 left ; what is his gross income ?

$$₨1 = 192p. ; ₨1 - 4p. = 188p.$$

$$\begin{array}{l} p. \quad p. \quad ₨. \\ 188 : 192 :: 4794 : x, \end{array}$$

i.e., $47 : 48 :: 4794 : x$;

$$\therefore \quad x = ₨\tfrac{48 \times 4794}{47} = ₨4896. \quad \textit{Ans.}$$

Example 5. If 8 oxen or 6 horses eat the grass of a field in 10 days, in how many days will 5 oxen and 4 horses eat it?

$$\begin{array}{ccccccc} \text{oxen} & & \text{oxen} & & \text{horses} & & \\ 8 & : & 5 & :: & 6 & : & x, \end{array}$$

$$\therefore \quad x = \tfrac{5\times6}{8} \text{ horses} = \tfrac{15}{4} \text{ horses}.$$

$\therefore$ 5 oxen and 4 horses will eat as much as $(\frac{15}{4}+4)$ or $\frac{31}{4}$ horses.

$$\begin{array}{lccccccc} & \text{horses} & & \text{horses} & & \text{days} & & \\ \text{Now,} & \frac{31}{4} & : & 6 & :: & 10 & : & x, \end{array}$$

$$\therefore \quad x = \tfrac{6\times10\times4}{31} \text{ days} = 7\tfrac{23}{31} \text{ days. } \quad \textit{Ans.}$$

Example 6. *A* can do a piece of work in 7 days, and *B* can do it in 9 days: how long will *A* and *B*, working together, take to do the work?

A can do $\frac{1}{7}$ of the work and *B* can do $\frac{1}{9}$ of the work in 1 day; $\therefore$ *A* and *B* together can do $(\frac{1}{7}+\frac{1}{9})$ or $\frac{16}{63}$ of the work in 1 day.

$$\begin{array}{ccccccc} \text{work} & & \text{work} & & \text{day} & & \\ \frac{16}{63} & : & 1 & :: & 1 & : & x, \end{array}$$

$$\therefore \quad x = \tfrac{63}{16} \text{ days} = 3\tfrac{15}{16} \text{ days. } \quad \textit{Ans.}$$

Example 7. At what time between 2 and 3 o'clock are the hands of a clock at right angles to each other?

The minute-hand gains 11 divisions on the hour-hand in 12 minutes; and here it has to gain $(10+15)$ or 25 divisions.

$$\begin{array}{ccccccc} \text{div.} & & \text{div.} & & \text{min.} & & \\ 11 & : & 25 & :: & 12 & : & x, \end{array}$$

$$\therefore \quad x = \tfrac{25\times12}{11} \text{ min.} = 27\tfrac{3}{11} \text{ min.};$$

$\therefore$ the two hands will be at right angles to each other at $27\frac{3}{11}$ minutes past 2.

Example 8. *A* can beat *B* by 40 yards in a mile race; *B* can beat *C* by 20 yards in a mile race; if *A* and *C* run a mile, by how much will *A* win?

While *A* runs 1760 yd., *B* runs 1720;
and .... *B* ...... 1760 yd., *C* ...... 1740.

$$\begin{array}{lccccccc} & 1760 & : & 1720 & :: & 1740 & : & x, \\ \textit{i.e.,} & 44 & : & 43 & :: & 1740 & : & x, \end{array}$$

$$\therefore \quad x = \tfrac{43\times1740}{44} \text{ yd.} = 1700\tfrac{5}{11} \text{ yd.}$$

$\therefore$ While *B* runs 1720 yd., *C* runs $1700\frac{5}{11}$ yd.; but while *B* runs 1720 yd., *A* runs 1760 yd.; $\therefore$ while *A* runs 1760 yd., *C* runs $1700\frac{5}{11}$ yd. $\therefore$ *A* will win by $(1760-1700\frac{5}{11})$ or $59\frac{6}{11}$ yd.

Example. 9. A starts from P to walk to Q, a distance of $51\frac{3}{4}$ miles, at the rate of $3\frac{3}{4}$ miles an hour; an hour later B starts from Q for P and walks at the rate of $4\frac{1}{4}$ miles an hour: when and where will A meet B?

A has already gone $3\frac{3}{4}$ miles when B starts. Of the remaining 48 miles, A walks $3\frac{3}{4}$ and B walks $4\frac{1}{4}$ in one hour; that is, they together pass over $(3\frac{3}{4}+4\frac{1}{4})$ or 8 miles in one hour.

$$\begin{array}{ccccccc} \text{miles} & & \text{miles} & & \text{hour} & & \\ 8 & : & 48 & :: & 1 & : & x, \end{array}$$

$$\therefore \quad x = \tfrac{48}{8} \text{ hours} = 6 \text{ hours.}$$

$\therefore$ A meets B in 6 hours after B started. And therefore they meet at a distance of $4\frac{1}{4}\times 6$ or $25\frac{1}{2}$ miles from Q.

[For Examples for Exercise see Section XXXV.]

XXXIX. DOUBLE RULE OF THREE.

220. Complex problems which would require two or more applications of the Rule of Three are usually solved by a shorter method, commonly called the **Double Rule of Three.** The method will be best explained by means of examples.

Example 1. If 9 men can reap 6 acres in 10 days, how many men will reap 12 acres in 15 days?

$$\left.\begin{array}{lccc} \text{acres} & 6 & : & 12 \\ \text{days} & 15 & : & 10 \end{array}\right\} :: 9 \text{ men} : x.$$

We denote the answer by x and place it for the 4th term, and place 9 men (which is of the same kind as the answer) for the 3rd term We next take 6 acres and 12 acres (a pair of quantities of the same kind), and consider whether the answer will be greater or less than the 3rd term in the question "if 9 men can reap 6 acres, how many men will reap 12 acres, supposing the time to be the same in both cases?" and we find that the answer will be greater; we therefore place 12 acres for the 2nd and 6 acres for the 1st term. Then we take 10 days and 15 days (another pair of quantities of the same kind), and consider whether the answer will be greater or less than the 3rd term in the question "if 9 men can reap in 10 days, how many men will reap in 15 days, supposing the number of acres to be the same in both cases?" and we find that the answer will be less; we therefore place 10 days for the 2nd and 15 days for the 1st term, under those already obtained. We now multiply the numbers in the 1st term for the final 1st term and the numbers in the 2nd term for the final 2nd term. Thus

$$6\times 15 : 12\times 10 :: 9 : x,$$

$$\therefore \quad x = \tfrac{12\times 10\times 9}{6\times 15} \text{ men} = 12 \text{ men.} \quad \textit{Ans.}$$

Note Each pair of quantities of the same kind should be replaced by their measures in terms of the same unit.

Remark. Each additional pair of quantities of the same kind would be treated in a like manner.

Example 2. If 72 men can dig a trench, 324 yd. long, 12 yd. wide and 8 ft. deep, in 9 days of 12 hours each; how many men can dig a trench, 1458 yd. long, 40 ft. wide and 3 yd. deep, in 36 days of 9 hours each?

$$\left.\begin{array}{lrcr} \text{ft. long} & 324\times3 & : & 1458\times3 \\ \text{ft. wide} & 12\times3 & : & 40 \\ \text{ft. deep} & 8 & : & 3\times3 \\ \text{days} & 36 & : & 9 \\ \text{hours} & 9 & : & 12 \end{array}\right\} :: 72 \text{ men} : x,$$

$$\therefore\ x=\frac{1458\times3\times40\times3\times3\times9\times12\times72}{324\times3\times12\times3\times8\times36\times9} \text{ men} = 135 \text{ men.} \quad \textit{Ans.}$$

Or better thus:

$$\left.\begin{array}{lll} \text{cu. ft.} & (324\times3)\times(12\times3)\times8 & :(1458\times3)\times40\times(3\times3) \\ \text{hours} & 36\times9 & :9\times12 \end{array}\right\} :: 72 : x.$$

Example 3. If 10 men can perform a piece of work in 24 days, how many men will perform another piece of work 3 times as great in $\frac{1}{5}$ of the time?

$$\left.\begin{array}{lrcr} \text{work} & 1 & : & 3 \\ \text{days} & \frac{24}{5} & : & 24 \end{array}\right\} :: 10 \text{ men} : x,$$

$$\therefore\ x=\frac{3\times24\times10}{\frac{24}{5}} \text{ men} = \frac{3\times24\times10\times5}{24} \text{ men} = 150 \text{ men.} \quad \textit{Ans.}$$

Example 4. If the sixpenny loaf weigh 8 oz. when wheat is 15*s*. a bushel, what ought a bushel of wheat to be when the fourpenny loaf weighs 12 oz.?

$$\left.\begin{array}{lrcr} \text{pence} & 6 & : & 4 \\ \text{ounces} & 12 & : & 8 \end{array}\right\} :: 15s. : x,$$

$$\therefore\ x=\frac{4\times8\times15}{6\times12}s.=\frac{20}{3}s=6s.\ 8d. \quad \textit{Ans.}$$

Example 5. If 5 cannon, which fire 3 rounds in 5 minutes, kill 135 men in $1\frac{1}{2}$ hours, how many cannon, which fire 5 rounds in 6 minutes, will kill 250 men in 1 hour?

[The first 5 cannon, each firing 54 rounds, kill 135 men: it is required to find how many cannon, each firing 50 rounds, will kill 250 men.]

$$\left.\begin{array}{lrcr} \text{rounds} & 50 & : & 54 \\ \text{men} & 135 & : & 250 \end{array}\right\} :: 5 \text{ cannon} : x,$$

$$\therefore\ x=\frac{54\times250\times5}{50\times135} \text{ cannon} = 10 \text{ cannon.} \quad \textit{Ans.}$$

221. Examples in Double Rule of Three can be worked more conveniently in a little different manner In this method the first *work* and second *work* are respectively taken for the third and fourth terms of the proportion, and the first *cause* and second *cause* respectively for the first and second terms; for, the ratio of the two *causes* is equal to the ratio of the corresponding *works*. We shall apply the method to the first two of the foregoing examples.

Example 1. 9 men in 10 days will do the same amount of work as (9×10) men will do in 1 day; and x men in 15 days will do the same amount of work as $(x\times15)$ men will do in 1 day.

$$\therefore\quad 9\times10 : x\times15 :: 6 : 12,$$
$$\therefore\quad x\times15\times6=9\times10\times12,$$
$$\therefore\quad x=\frac{9\times10\times12}{15\times6}\text{ men}=12\text{ men.}\quad \textit{Ans.}$$

Example 2.

$$72\times9\times12 : x\times36\times9 :: (324\times3)\times(12\times3)\times8 : (1458\times3)\times40\times(3\times3),$$
$$\therefore\quad x=\frac{72\times9\times12\times1458\times3\times40\times3\times3}{36\times9\times324\times3\times12\times3\times8}\text{ men}$$
$$=135\text{ men.}\quad \textit{Ans.}$$

[For Examples for Exercise see Section xxxvi.]

MISCELLANEOUS EXAMPLES. 139.

1. Find the least number which being added to 1409 will make the result divisible by 23.

2. A boy receiving ₨2. 4*a*. a week has 8*a*. stopped every fourth week; if there are 48 weeks in the school-year, how much does he get in 2 years?

3. What are the prime factors in 45090045, and what is the smallest whole number by which it must be multiplied in order to become a perfect square?

4. Find the least fraction which, being added to $\frac{1}{6}+\frac{1}{6}\div\frac{1}{7}-\frac{1}{7}\times\frac{1}{8}-\frac{1}{8}$, shall make the result an integer.

5. Find, by Practice, the value of $37\frac{1}{2}$ md. of sugar at ₨9. 13*a*. 6*p*. per md.

6. If 27 men can perform a piece of work in 15 days, how many men must be added to the number that the work may be finished in $\frac{3}{5}$ of the time?

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7. Find the greatest and least numbers of four digits exactly divisible by 34.

8. I distribute a sum of money among 32 men, giving ₨50. 7*a*. 6*p*. to the first, ₨51. 7*a*. 6*p*. to the next, ₨52. 7*a*. 6*p*. to the next,

and so on, increasing the sum by ₨1 each time; how much would each get if I divided the money equally?

9. Determine the least number, by which 378 must be multiplied to produce a number exactly divisible by 336.

10. A screw advances ·392 of an inch at each turn; how many turns must be taken for it to advance 9·8 inches?

11. Find, by Practice, the cost of 35 cwt. 2 qr. 7 lb. at £7. 11*s.* 4*d.* per cwt.

12. If 12 iron bars, each 4 ft. long, 3 in. broad and 2 in. thick, weigh 576 lb., how much will 11 weigh, each 6 ft. long, 4 in. broad and 3 in. thick?

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13. The population of a town is 5720, and there are 320 more men than women; how many are there of each sex?

14. A labourer, who works on week days only, earns 7*a.* 9*p.* a day; supposing that the 1st of January 1885 was on a Sunday, find the amount of his earnings during the year.

15. Four bells ring at intervals of 3, $3\frac{1}{4}$, $3\frac{1}{2}$ and $3\frac{3}{4}$ seconds respectively, beginning together; how often during 24 hours will the four bells ring together again?

16. By what number must $\frac{1}{2}+\frac{1}{3}$ of $\frac{1}{6}-\frac{1}{4}$ be multiplied in order to produce the least possible integer?

17. A certain number of men subscribed £63. 0*s.* 9*d.*, each subscribing as many pence as there were men; how many men were there?

18. If $\cdot\dot{4}2857\dot{1}$ of a barrel of beer be worth ·72 of £2. 10*s.*, what is the value of ·625 of the remainder?

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19. To the fourth part of a certain number I add 79, and obtain 100 as the sum; what is the number?

20. Divide ₨101. 15*a.* 3*p.* among 20 men, giving to each of 5 of them twice as much as to each of the others.

21. 720 gallons of cocoanut oil and 450 gallons of castor oil are to be put into an exact number of barrels, all of the same size, without mixing the two oils together; find the least number of barrels required.

22. Express $\frac{3}{8}$ of 7*s.* 6*d.* + 1·25 of 5*s.* − $\cdot 5\dot{4}\dot{5}$ of 9*s.* 2*d.* as the decimal of £10.

23. The perimeter of a rectangle is 110 ft.; the difference of two sides is 11 ft.: find its area as the decimal of an acre.

24. If a man can perform a journey of 170 miles in $4\frac{1}{4}$ days of 11 hours each, in how many days of $8\frac{3}{4}$ hours each, will he perform a journey of 470 miles?

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25. To a certain number I add 3, and multiply the sum by 4, then divide the product by 5, and get 7 as quotient and 1 as remainder: what is the number?

26. A man bought 40 pieces of ribbon, all equally long, for R 137. 8*a.* at 2*a.* 9*p.* a yard; how many inches were there in each piece?

27. What is the least debt in dollars (4*s.* 2*d.* each) that can be paid in moidores?

28. What is the capacity of a vessel, out of which, when it is half full, $4\frac{1}{2}$ gallons being drawn, there remains $\frac{1}{6}$ of the whole content?

29. A square space, containing 113 sq. yd. 7 sq. ft., is to be lengthened by 3 ft. in one of its dimensions, and to be shortened by 3 ft. in the other; what will then its area be?

30. If a person walks 7 miles in $2\frac{1}{2}$ hours, how long will a second person take to walk 10 miles, supposing that the first walks $2\frac{3}{4}$ miles while the second walks $2\frac{1}{4}$?

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31. Fourteen years ago a man was six times as old as his son whose present age is 20 years; what is the present age of the father?

32. A man buys 20 seers of milk at 3*a.* 6*p.* per seer; how much water must he add to it that he may gain R1. 4*a.* by selling the mixture at 3*a.* per seer?

33. I had coins of one kind weighing 2295 grains; and of this I spent coins weighing 1035 grains: show that a single coin cannot weigh more than 45 grains.

34. Two clocks begin to strike 12 together: one strikes at an interval of $2{\cdot}91\dot{6}$ seconds, the other, of $2{\cdot}08\dot{3}$ seconds; what decimal of a minute is there between their seventh strokes?

35. Find the cost of painting the walls of a square room, 10 ft. high and 16 ft. long, with one door 8 ft. by 4 ft., and 2 windows, each 5 ft. by 2, the amount saved by each window being R1. 14*a.* What additional height would increase the cost by R12?

36. A merchant of Calcutta indented from London goods worth £226, and paid £34 for freight and packing. He sold half the goods at a gain of 2 annas per rupee; at what gain per rupee must he sell the remainder that he may clear R500 on the whole outlay? [R1 = 1*s.* $7\frac{1}{2}$*d.*]

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37. Find the greatest fraction, the numerator of which is composed of 3, 5, 1, 0 and the denominator of 3, 2, 8, 0.

38. Two persons buy 600 oranges each at 24 for a half-rupee;

one sells them at 5*a*. 6*p*. a dozen, and the other at 8*a*. 3*p*. a score who gains more, and by how much ?

39. A number is exactly divisible by 7 and by 13, and it known that the number is between 400 and 500; what is the numbe

40. What fraction of $\frac{9}{4}$ of a rupee is $\frac{4}{5}$ of R5; and wh fraction of their sum is their difference ?

41. Find the length of the inner edge of a cubical cister which will hold 256 lb. of water, supposing that a cu. ft. of wat weighs 1000 oz.

42. A person after paying an income-tax of 1 anna in the R devotes $\frac{1}{16}$ of the remainder of his income to purposes of charit and finds that he has R5175 left ; what is his income ?

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43. A person has a number of oranges to dispose of ; he sel half of what he has and one more to *A*, half of the remainder an one more to *B*, half of the remainder and one more to *C* ; by whic time he has disposed of all he had : how many had he at first ?

44. A certain number of men, twice as many women and thre times as many children earned R16. 2*a*. in 3 days ; each ma earned 12*a*., each woman 8*a*. and each child 5*a* a day : how man women were there ?

45. Find the greatest weight that will measure (*i.e.*, divid exactly) a lb. avoir, and a lb. troy.

46. If $\frac{13}{18}$ of a number exceed $\cdot 8\dot{3}$ of half the number by $\cdot 200$ what must the number be ?

47. How many bricks, 6 in. by 3 in. by 3 in., will be require for a wall, 16 ft. by 10 ft. by 2 ft., allowing $\frac{1}{10}$ of the space fo mortar ?

48. A creditor received on a debt of R3600 a dividend o 9*a*. 10*p*. in the R ; and a further dividend of 6*a*. 8*p*. upon th remainder. What did he receive altogether, and what fractio was it of the entire debt ?

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49. *A* has R150, *B* has R120 ; if *C* has R16 more than wha he has, then *B* and *C* together would have as much as *A* : how much has *C* ?

50. Divide £30. 10*s*. 8*d*. into two sums of money, one of whic contains as many shillings as the other contains fourpences.

51. 378 oranges and 462 mangoes are to be distributed among boys so that each boy gets as many oranges and as many mangoe as any other boy ; find the largest possible number of boys, an the least possible number of fruits each boy may get.

52. What number is greater than its fifth part by $\frac{1}{6}$?

53. Find how much card-board is required to make a cubical box and its cover, the edge of the box is 9 in., and the rim of the cover extends 3 in. deep down each side.

54. A work can be completed in 36 days by 30 men working 6 hours a day; in what time would 18 men and 60 women, working 9 hours a day, complete it, supposing that 3 men can do as much work as 5 women?

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55. A gentleman's monthly expenses are ₨150 less than his income; if his income be increased by ₨100 a month and expenses decreased by ₨50, how much will he be able to save in a year?

56. Three persons *A*, *B*, *C* start on a tour, each with £20 in his pocket, and agree to divide their expenses equally. When they return, *A* has £3. 11*s*. 9*d*., *B* has £2. 5*s*. and *C* has 17*s*. 3*d*. What ought *A* and *B* to pay to *C* to settle their accounts?

57. A man walks at the rate of 128 yards per minute; find the least whole number of minutes he will take to walk over an exact number of miles.

58. Simplify $(3\cdot5-2\cdot\dot{3})(3\cdot5+2\cdot\dot{3})\div3\cdot5$ of $2\cdot\dot{3}\times32\cdot5\dot{3}$.

59. The external dimensions of an open box are 5 ft., $4\frac{1}{2}$ ft. and 3 ft.; find the cost of painting the outside at 3 annas per sq. yd. What will be the cost of painting the inside at the same rate, if the box is made of $\frac{1}{2}$-inch plank?

60. Three men can do as much work as 5 boys; the wages of three boys are equal to those of two men. A work, on which 40 boys and 15 men are employed, takes 8 weeks and costs £350; how long would it take if 20 boys and 20 men were employed, and how much would it cost?

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61. What quantity of water must an inn-keeper add to a barrel of beer, which cost him £50, to reduce the price to £1. 5*s*. a gallon?

62. A certain number of men mow 4 acres in 3 hours, and a certain number of others mow 8 acres in 5 hours; how long will they be mowing 11 acres, if they all work together?

63. At 10 minutes to 2 in the afternoon a clock is 55 seconds slow, and at 6 in the evening it is 30 seconds slow; at what hour will it show true time?

64. A train leaves Calcutta at 7 A. M. for Goalundo, 153 miles distant, and travels at the rate of 20 miles an hour; another train leaves Goalundo for Calcutta at 11-30 A. M. and travels at the rate of 22 miles an hour; when, and where, will the trains pass each other?

65. A cistern, 6 ft. long, 5 ft. wide and 4 ft. deep, contains

pulp for making paper. If $\frac{2}{3}$ of the volume of the pulp be lost in the process of drying, how many sheets of paper, 16 in. by 10 in., will be obtained, if 400 sheets in thickness go to an inch?

66. If 7 men and 5 boys can reap 168 acres in 18 days, how many days will 15 men and 5 boys take to reap 700 acres, one man being able to do three times as much work as a boy?

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67. Find the value of $\frac{3}{8}$ of a guinea $+\frac{6}{11}$ of 8*s.* 3*d.* $+\frac{3}{110}$ of £2. 15*s.*; and reduce the result to the fraction of a guinea and a half.

68. Two pipes, *A* and *B* fill a cistern in 25 and 30 minutes respectively. Both pipes being opened, find when the first must be closed that the cistern may be just filled in 15 minutes.

69. If $\frac{1}{5}$ of a sheep be worth $\frac{2}{3}$ of a rupee, and $\frac{2}{7}$ of a sheep be worth $\frac{1}{14}$ of a cow, how much must be given for 106 cows?

70. The cubic content of an open cistern, 6 ft. long and 4 ft. broad, is 20 cu. ft.; what will be the cost of lining the inside of it at 1*s.* per sq. ft.?

71. Two persons walking at the rate of $3\frac{1}{2}$ and 4 miles per hour respectively, set off from the same place in opposite directions to walk round a park, and meet in 20 minutes. Find the length of the path round the park.

72. If it takes 120 men to supply, in 5 days' work, a fortress with provisions for 5 months, when the garrison is 650 strong, how many will be required to supply it in 3 days for 4 months, after the garrison has been reduced by 130 men?

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73. A bag contains a certain number of shillings, twice as many sixpenny pieces and 3 times as many fourpenny pieces; the whole sum amounts to 2 guineas; find the number of each.

74. A room, whose height is 9 ft., and length twice its breadth, takes 189 yards of paper, 2 ft. wide, for its four walls; find its length.

75. *A* can do a piece of work in 20 days; *A* and *B* together can do it in $11\frac{1}{9}$ days. *A* works alone for 8 days, *A* and *C* together for 6 days, and *B* finishes it in 3 days. Find in what time *B* and *C* together could do it.

76. One clock gains 8 min., and another loses 4 min., in 24 hours. They are set at right at noon on Sunday. Determine the time indicated by each clock when the one appears to have gained 12 minutes on the other.

77. The whole time occupied by a train 110 yards long, travelling at the rate of 30 miles an hour, in crossing a bridge is 12 seconds; find the length of the bridge.

78. If a family of 9 persons spend ₨480 in 8 months, how much will serve a family (living upon the same scale) of 24 persons for 16 months?

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79. Simplify $\dfrac{£7.\ 6s.\ 8d.}{£3.\ 4s.} \times \dfrac{\frac{1}{2}-\frac{1}{3} \text{ of } \frac{1}{4}-\frac{1}{5}}{(\frac{1}{2}-\frac{1}{3}) \text{ of } (\frac{1}{4}-\frac{1}{5})}$.

80. A room twice as long as it is broad is carpeted at 9*s.* a sq. yd., and the walls are painted at 1*s.* 6*d.* a sq. yd., the respective costs being £44. 2*s.* and £8. 8*s.* Find the dimensions of the room.

81. A cistern would be filled by a tap, *A*, in $3\frac{1}{2}$ hours, or emptied by a tap, *B*, in 3 hours. The cistern being half full, *A* is turned on at 8 o'clock, and *B* at 15 min. to 9; find when the cistern will again be half full.

82. If 2 guineas make 3 napoleons, and 15 rix-dollars make 4 napoleons, and 6 ducats make 7 rix-dollars, how many ducats are there in £490?

83. A person rows a distance of 3 miles down a stream in 40 minutes, but without the aid of the stream it would have taken him an hour; what is the rate of the stream per hour? and how long would it take him to return against it?

84. A boat propelled by 6 oars which take 25 strokes per minute travels at the rate of $7\frac{1}{2}$ miles an hour; find the rate of a boat propelled by 4 oars which take 32 strokes per minute; the work done by each oar during one stroke in the latter case being a quarter as much again as in the former case.

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85. A wagon, loaded with 1246 equal packages, weighs 26 tons 14 cwt.; if the wagon itself weighs twice as much as the packages, find the weight of each package.

86. *A* did $\frac{3}{5}$ of a piece of work in 6 hours, *B* did $\frac{3}{4}$ of what remained in 2 hours and *C* finished it in half an hour. How long would they have been doing the whole if they had worked together?

87. A clock loses 5 minutes a day. It shows correct time at noon on a Monday. After how many days will it again show correct time on a Monday?

88. A privateer, running at the rate of 10 miles an hour, discovers a ship, 18 miles off, making way at the rate of 8 miles an hour; how many miles can the ship run before she is overtaken?

89. If the wages of 25 men amount to ₨766. 10*a.* 8*p.* in 16 days, how many men must work 24 days to receive ₨1035, the daily wages of the latter being one-half those of the former?

90. 55 gallons of a mixture of wine and water contain 5 gallons more wine than water; find the ratio of wine to water in the mixture.

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91. Bring $\left\{\left(\frac{5\frac{1}{4}-\frac{1}{3}\text{ of }2\frac{5}{8}}{\frac{3}{8}\times 4\frac{1}{6}+\frac{1}{12}}+\frac{2\frac{3}{7}}{4\frac{3}{7}}\right)\div 21\frac{28}{29}\times 3\frac{19}{206}\right\}$ cwt. to the fraction of $4\frac{1}{7}$ tons.

92. *A* can do half a piece of work in 3 hours, being twice as much as *B* can do; *A*, *B* and *C* can together do the whole in $2\frac{1}{3}$ hours: in how many hours will *C* do a piece of work which *B* can do in 9 hours?

93. How many seconds will a train, 184 feet in length, travelling at the rate of 21 miles an hour, take in passing another train, 223 feet long, proceeding in the same direction at the rate of 16 miles an hour?

94. *A* can give *B* 20 yards' start in a mile race and can give *C* 40 yards' start; how much start can *B* give *C* in a mile race?

95. A piece of work must be finished in 36 days, and 15 men are set to do it, working 9 hours a day; but after 24 days it is found that only $\frac{3}{8}$ of the work is done. If 3 additional men be then put on, how many hours a day will they all have to labour, in order to finish the work in time?

96. Two equal wine glasses are filled with mixture of wine and water in the ratios of 2 of wine to 3 of water and 3 of wine to 4 of water; when the contents are mixed in a tumbler, find the strength of the mixture.

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97. Divide ₨47 between *A*, *B* and *C* in such a manner that *B* may receive ₨2 more than 3 times, and *C* ₨3 more than 4 times the amount to be received by *A*.

98. At what times between 2 and 3 are the hands of a clock $5\frac{1}{2}$ minute-divisions apart?

99. Three boys agree to start together and run, until all come together again, round a circular court 15 yards in circumference. One runs at the rate of 6, the second, 7, and the third, 8 miles an hour. In how many seconds will the race end?

100. In a game of skill *A* can give *B*, and *B* can give *C*, 10 points out of a game of 50; how many should *A* give *C*?

101. If 7 cows and 20 sheep be worth £12, and 3 cows and 16 sheep be worth £7, find the price of a cow and of a sheep.

102. Two equal wine glasses are respectively $\frac{1}{3}$ and $\frac{1}{4}$ full of wine; they are then filled up with water, and the contents mixed in a tumbler: find the ratio of wine to water in the tumbler.

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103. Express $\dot{6}$ of ₨17. 8*a.* + $\dot{5}$ of £1. 14*s.* 6*d.* as the fraction of ₨170, a rupee being worth 2 shillings.

104. *A* can do a piece of work in 8 days, which *B* can destroy in 3. *A* has worked 6 days, during the last 2 of which *B* has been destroying; how many days must *A* now work alone in order to complete his task?

105. A train 110 yd. long overtook a person walking along the line at the rate of 3 miles an hour, and passed him completely in 9 seconds; afterwards it overtook another person and passed him in $9\frac{3}{8}$ seconds. At what rate was this second person walking?

106. In a hundred yards' race *A* can give *B* four and *C* five yards' start: if *B* were to race *C*, giving him 1 yard in a hundred, which would win?

107. If 6 men and 2 boys can reap 13 acres in 2 days, and 7 men and 5 boys can reap 33 acres in 4 days, how long will it take 2 men and 2 boys to reap 10 acres?

108. Gold and silver are mixed together in a mass of 30 oz., so that for every 6 parts of gold there are 4 parts of silver. How much gold must be added to the mass, so that for every 5 parts of gold there may be 3 parts of silver?

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109. A publican bought 10 gallons of wine at £1.7.6 per gallon; he mixed some water and filled quart bottles with it: how much water must have been added, supposing that the cost price of the contents of each bottle was thereby reduced to 5*s.* $8\frac{3}{4}$*d.*?

110. If 12 oxen be worth 29 sheep, 15 sheep worth 25 hogs, 17 hogs worth 3 loads of wheat, and 8 loads of wheat worth 13 loads of barley; how many loads of barley must be given for 340 oxen?

111. *A* and *B* are two spouts attached to a cistern. *A* can fill it in 10 min., and *B* can empty it in 15 min. If *A* and *B* be opened alternately for 1 minute each, in what time will the cistern be filled?

112. A race course is one mile long: *A* and *B* run a race and *A* wins by 80 yards; *A* and *C* run over the same course and *A* wins by 20 seconds; *B* and *C* run and *B* wins by 5 seconds. In what time can *A* run a mile?

113. If I can walk a certain distance in 112 days when I rest 5 hours each day, how long will it take me to walk twice as far, if I walk twice as fast and rest twice as long each day?

114. A cask contains 12 gallons of a mixture of wine and water in the ratio of 3 to 1; how much of the mixture must be drawn off, and water substituted for the mixture in the cask to become half and half?

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115. A rectangular court is 50 yards long and 30 yards broad. It has paths joining the middle points of the opposite sides, of 6 ft. in breadth, and also has within it a path of the same breadth running all round it. The remainder is covered with grass. If the cost of the pavement be 1*s.* 8*d.* per sq. ft., and of the grass 3*s.* per sq. yd., find the whole cost of laying out the court.

116. To complete a piece of work, *A* would take twice as long as *B* and *C* together, and *B* 3 times as long as *A* and *C* together; *A*, *B*, *C* together can do it in 12 days. In what time could each do it by himself?

117. A down-train usually travels at the rate of 30 miles an hour and meets an up-train 50 miles from the terminus. On one occasion, on account of an accident, it only went at the rate of 20 miles an hour and met the up-train $41\frac{2}{3}$ miles from the terminus. Find the speed of the up-train.

118. *A* can walk 5 miles an hour, and the rates at which *A* and *B* walk are in the ratio of 7 to 6: how many seconds' start must *A* give *B* that he may just beat him in a 3-mile race?

119. If 5 pumps, each having a length of stroke of 3 feet, working 15 hours a day for 5 days, empty the water out of a mine; how many pumps with a length of stroke of $2\frac{1}{2}$ feet, working 10 hours a day for 12 days, will be required to empty the same mine; the strokes of the former set of pumps being performed four times as fast as the other?

120. If 7 horses and 12 cows cost as much as 10 horses and 6 cows, compare the prices of a horse and a cow.

XL. DIVISION INTO PROPORTIONAL PARTS.

222. To divide a given quantity into *proportional parts* is to divide it into parts which shall be proportional to certain given numbers.

Example 1. Divide R873 among *A*, *B*, *C*, so that their shares may be in the proportion of 2, 3 and 4.

If we divide R873 into 9 (*i.e.*, 2+3+4) equal parts, then *A* will have 2, *B* will have 3 and *C* will have 4 of these parts.

Hence *A*'s share $= \text{R}\frac{873}{9} \times 2 = \text{R}194$

B's share $= \text{R}\frac{873}{9} \times 3 = \text{R}291.$

C's share $= \text{R}\frac{873}{9} \times 4 = \text{R}388.$

Example 2. Divide £287 into parts proportional to $1\frac{1}{2}$, 2 and $3\frac{1}{3}$

$$1\tfrac{1}{2} : 2 : 3\tfrac{1}{3} = \tfrac{3}{2} : 2 : \tfrac{10}{3} = \tfrac{9}{6} : \tfrac{12}{6} : \tfrac{20}{6} = 9 : 12 : 20.$$

Now proceed as in the preceding example.

Example 3. A certain sum of money was divided between A, B, C in the proportion of 5, 6 and 9; A received £45; what was the sum divided?

Since $5+6+9=20$, if the whole sum were divided into 20 equal parts, A's share would contain 5 of these parts. Hence the value of one part $=£\frac{45}{5}$; $\therefore$ the whole sum $=£\frac{45}{5}\times 20=£180$.

Example 4. Divide ₨50 among A, B, C, so that B's share may be half as much again as A's, and C's share $\frac{2}{3}$ of A's and B's together.

$$B\text{'s share}=1\tfrac{1}{2}\text{ of }A\text{'s share};$$

$$\therefore\ A\text{'s share}+B\text{'s share}=A\text{'s share}+1\tfrac{1}{2}\text{ of }A\text{'s share}$$
$$=(1+1\tfrac{1}{2})\text{ of }A\text{'s share}=2\tfrac{1}{2}\text{ of }A\text{'s share};$$

$$\therefore\ C\text{'s share}=\tfrac{2}{3}\text{ of }2\tfrac{1}{2}\text{ of }A\text{'s share}=\tfrac{5}{3}\text{ of }A\text{'s share};$$

$$\therefore\ A\text{'s share} : B\text{'s} : C\text{'s} = 1 : 1\tfrac{1}{2} : \tfrac{5}{3}\text{; etc.}$$

Example 5. Divide 52 into 3 parts such that $\frac{1}{3}$ of the first part $=\frac{1}{2}$ of the second part $=5$ times the third part.

$$\tfrac{1}{2}\text{ of the 2nd part}=\tfrac{1}{3}\text{ of the 1st part},$$
$$\therefore\ \text{the 2nd part}=\tfrac{2}{3}\text{ of the 1st part}.$$
$$\text{Again, 5 times the 3rd part}=\tfrac{1}{3}\text{ of the 1st part},$$
$$\therefore\ \text{the 3rd part}=\tfrac{1}{15}\text{ of the 1st part}.$$
$$\therefore\ \text{1st part} : \text{2nd part} : \text{3rd part}$$
$$=\text{1st part} : \tfrac{2}{3}\text{ of the 1st part} : \tfrac{1}{15}\text{ of the 1st part}$$
$$=1 : \tfrac{2}{3} : \tfrac{1}{15}\text{; etc.}$$

Example 6. ₨82 is given to 5 men, 8 women and 10 boys, in such a way that a woman is to receive twice as much as a boy, and a man as much as a woman and a boy together; what do the women receive?

8 women receive as much as 16 boys;

and 5 men receive as much as 5 women and 5 boys,

or as 10 boys and 5 boys,

or as 15 boys;

$\therefore$ men's share : women's : boys' $=15 : 16 : 10$; etc.

Example 7. How many rupees, half-rupees and quarter-rupees of which the numbers are proportional to 3, 4 and 5 are together worth ₨50?

Value of three groups of coins are

as 3 rupees : 4 half-rupees : 5 quarter-rupees,

or as 12 quarter-rupees : 8 quarter-rupees : 5 quarter-rupees,

or as 12 : 8 : 5 ;

$\therefore$ the amount in rupees $=$ R$\frac{50}{25}\times 12=$ R24 ;

the amount in half-rupees $=$ R$\frac{50}{25}\times 8=$ R16 ;

and the amount in qr.-rupees $=$ R$\frac{50}{25}\times 5=$ R10.

Therefore there are 24 rupees, 32 half-rupees and 40 qr.-rupees.

Example 8. Divide £100 between *A*, *B*, *C*, *D*, so that *A*'s share : *B*'s $=2:3$, *B*'s : *C*'s $=4:5$ and *C*'s : *D*'s $=7:8$.

We find as in *Ex.* 4, Art. 216, that the shares of *A*, *B*, *C*, *D* are proportional to 56, 84, 105 and 120 ; etc.

EXAMPLES. 140.

1. Divide R15. 10*a*. into parts proportional to 1, 2, 3, 4.

2. Divide £18. 9*s*. into parts proportional to 3, $2\frac{1}{2}$, 1, $\frac{1}{2}$.

3. Divide 26 tons in the proportion of 3·5, 2·25, $3\frac{1}{3}$, $3\frac{1}{12}$.

4. Divide $532\frac{1}{2}$ into parts which shall have the same ratio to one another as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$.

5. Divide £4. 17*s*. 6*d*. into two parts one of which is $\frac{5}{8}$ of the other.

6. A sum of money was divided into parts proportional to $3\frac{3}{4}$, 4, 5·5 ; the smallest part was R30 ; what was the sum divided ?

7. A sum of money was divided between *A*, *B*, *C*, in proportion to their ages which were 10, 12, 13 years respectively ; *A*'s share was £55 ; find the other shares.

8. Gunpowder is composed of saltpetre, sulphur and charcoal, in parts proportional to 75, 10 and 15 ; how many pounds of charcoal are there in 6 cwt. of gunpowder ?

9. How much of the above gunpowder can be made with 25 lb. of sulphur ?

10. In a certain battle an army lost 4 men wounded and 2 killed out of every 25, and it mustered 38,000 men unhurt ; what was the number of men in the army at first ?

11. Divide R90 between three persons, so that for every rupee given to the first, the second may get 12 annas and the third may get 8 annas.

12. Divide R36 between *A*, *B* and *C*, so that *A* gets $\frac{3}{4}$ of *B*'s share, and *C* gets $\frac{2}{3}$ of *A*'s share.

13. Divide ₨360 among A, B, C, so that A may get 3 times as much as B, and B and C together $\frac{1}{2}$ as much as A.

14. Divide ₨32 between A, B, C, so that A may receive 3 times as much as B, and C $\frac{1}{3}$ of what A and B together receive.

15. Divide £14 between A and B, so that $\frac{1}{2}$ of A's money may be equal to $\frac{2}{3}$ of B's.

16. Divide 30 into 3 parts such that $\frac{1}{3}$ of the first part $=\frac{2}{5}$ of the second $=\frac{1}{2}$ of the third.

17. ₨21 is divided between A, B, C; A's share is $\frac{2}{5}$ of B's; it is also $\frac{2}{5}$ of B's and C's together; find each one's share.

18. Divide £1. 13*s*. $4\frac{1}{2}$*d*. between A, B, C, D, so that A's share may be $\frac{3}{10}$ of D's, C's share $\frac{3}{10}$ of A's, and B's share the sum of A's and C's.

19. Divide £3. 6*s*. between 5 men, 7 women and 10 boys, so that each woman may have $\frac{3}{5}$ of each man's share, and each boy $\frac{2}{3}$ of each woman's share.

20. ₨110 is to be divided among 10 men, 16 women and 20 children: if each man's share is to be equal to the shares of 2 women and the 16 women are to have twice as much as the 20 children, how much will each woman receive?

21. A number of men, women and children are in the proportion of 3, 4, 5; divide £3. 5*s*. 3*d*. among them, so that the shares of a man, a woman and a child may be proportional to 4, 3, 1.

22. Divide £39 among A, B, C, so that A's share : B's share $=3:2$, B's share : C's share $=4:3$.

23. A certain kind of brass is composed of copper, zinc, lead and tin: the ratio of copper to zinc is $1:2$, of zinc to lead $3:5$ and of lead to tin $7:8$; find the quantity of zinc in 1 cwt. of the brass.

24. Four towns are to provide according to their population a contingent of 140 men. The populations of the towns are 1058, 1587, 2116 and 2645 respectively; find the number of men to be provided by each town.

25. 700 coins consist of rupees, half-rupees and quarter-rupees, the values of the rupees, the half-rupees and the quarter-rupees are as $2:3:5$; find the number of the rupees.

26. How many rupees, eight-anna pieces and four-anna pieces, of which the numbers are proportional to $2\frac{1}{2}$, 3 and 4, are together worth ₨80?

27. If 2 men do as much work as 5 women, and 6 women as much as 10 children, divide a week's wages of ₨38 among 8 men, 9 women and 15 children.

28. The sum of three fractions is $\frac{122}{105}$; 14 times the first = 15 times the second = 18 times the third : find the fractions.

29. Divide ₨142 among *A*, *B*, *C*, so that for every ₨5 given to *A*, *B* may get ₨3, and for every ₨7 given to *B*, *C* may get ₨5.

30. Areas of circles are to one another as the squares of their radii. Divide a circle of 1 ft. radius into three equal parts by concentric circles.

31. If the weight of pure silver and of alloy in a rupee be in the ratio of 11 to 1, and the price of pure silver be ₨2 10. $5\frac{1}{11}$ per oz. avoir., find the weight of a rupee (in grains) supposing its value to be that of pure silver it contains.

32. An estate is divided amongst 3 persons in the ratio of 7, 8 and 10. Find the value of the estate when ₨2500 added to the largest share would make it equal to half of the whole.

33. A number of mangoes is to be divided amongst 4 persons in shares which are as $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{1}{7}$: what must the number *at least* be that this may be done without cutting any of the mangoes.

XLI. FELLOWSHIP OR PARTNERSHIP.

223. Suppose that three persons *A*, *B*, *C*, are partners in trading, and that *A* has a capital of ₨3000 in the business, *B* has ₨5000 and *C* has ₨6000; and they gain ₨1400 : how should the profit be divided ?

It is obvious that the profit must be divided into parts proportional to 3000, 5000 and 6000; which may be done by the method explained in the preceding section.

The above is an example of what is called **Simple Fellowship**, the capitals contributed by the several partners being supposed to continue in the business for the same period of time.

224. Suppose, again, that *A*, *B*, *C*, are partners in trading, and that *A* has a capital of ₨3000 in the business for 3 months, *B* has ₨5000 for 6 months, and *C* has ₨6000 for 7 months, at the end of which time the gain is ₨720 : how should this be divided ?

Now, a capital of ₨3000 employed for 3 months may be taken to be equivalent to a capital of ₨9000 (*i.e.*, ₨3000 × 3) employed for 1 month; ₨5000 for 6 months may be taken to be equivalent to ₨30000 (*i. e.*, ₨5000 × 6) for 1 month; and ₨6000 for 7 months, equivalent to ₨42000 (*i. e.*, ₨6000 × 7) for 1 month. Hence the profit must be divided in the proportion of 9000, 30000 and 42000; which may be done in the usual way.

Consequently, if the capitals of partners be employed for different periods of time, the period of time must be made the same

for all, by multiplying each capital by the measure of the corresponding period of time.

Note. In working, the several sums of money must be expressed in terms of the same unit, as also the several periods of time.

The above is an example of what is called **Compound Fellowship**, the capitals contributed by the several partners being employed in the business for different periods of time.

EXAMPLES. 141.

1. *A*, *B*, *C* enter into partnership; *A* furnishes ₨350, *B* ₨500 and *C* ₨750; what should be the share of each in ₨320 profit?

2. A bankrupt owes ₨2000 to two creditors, namely, ₨1200 to one, and ₨800 to the other; his assets are ₨700; what does each creditor lose?

3. *A*, *B*, *C*, *D* engage in business with a joint capital of £7550; at the end of a year *A* receives £200, *B* £235, *C* £120 and *D* £200; how much capital did *C* put in?

4. *A*, *B*, *C* are partners, *A* receiving $\frac{2}{3}$ of the profits, and *B* and *C* sharing the remainder equally. *A*'s income is increased by ₨75 when the profits rise from $\frac{1}{12}$ of the capital to $\frac{1}{10}$. Find the respective capitals invested.

5. *A* and *B* are partners in a business in which *A* has $7\frac{1}{2}$ annas share and *B* $8\frac{1}{2}$ annas; *B*, being the working partner, receives $\frac{1}{20}$ of all the profit; the rest is divided in proportion to the capital; what does *B* receive out of ₨6080?

6. *A*, *B*, *C* engage in business with a joint capital of £18000; *A* gives £2000 more than *B*, and *B* £2000 more than *C*; divide a profit of £1080 between them.

7. *A*, *B*, *C* enter into partnership; *A* puts in £70 for 5 months, *B* £50 for 6 months and *C* £30 for 8 months. They gain £44. 10*s*. How should the profit be divided?

8. *A*, *B*, *C* pasture in the same field. *A* has in it 10 oxen for 7 months, *B* has 12 oxen for 5 months, and *C* has 15 oxen for 3 months. The rent is ₨17. 8*a*. How much of the rent should each pay?

9. *A* starts business with a capital of £2200 on the 16th of April, and on the 3rd of July admits a partner *B* with a capital of £1800. The profits amount to £449. 16*s*. by the 31st of December. What is each person's share?

10. *D* and *E* become partners, *D* bringing ₨5400 and *E* ₨4500. At the end of 3 months *D* doubles his capital and a new partner *F* is admitted who brings ₨5700; and at the end of

5 months E trebles his capital. The year's profits amount to ₹1200 : how ought this to be divided ?

11. A and B start a business with capitals as 5 : 7. They withdraw respectively $\frac{2}{3}$ and $\frac{3}{4}$ of their capitals at the end of 4 months. At the end of the year a profit of £226 is divided ; find A's share.

12. A and B entered into partnership with £700 and £600 respectively. After 3 months A withdrew $\frac{2}{7}$ of his stock but after 3 months more he put back $\frac{3}{8}$ of what he had withdrawn. The profits at the end of the year are £726 : how much of this should A receive ?

13. A and B start a business ? A puts in double of what B puts. A withdraws $\frac{1}{3}$ of his stock at the end of 3 months but at the end of 7 months puts back $\frac{1}{5}$ of what he has taken out, when B takes out $\frac{1}{4}$ of his stock. A receives ₹300 profits at the end of the year ; what does B receive ?

14. A and B hire a meadow for 6 months. A puts in 21 cows for 4 months ; how many can B put in for the remaining 2 months, if he pays $\frac{5}{7}$ of what A pays ?

XLII. ALLIGATION.

225. The following are examples of **Alligation** or the mixing of things of the same kind but of different qualities.

Example 1. How must a grocer mix teas at 2*s.* 6*d.* a lb. and 3*s.* 9*d.* a lb. so that the mixture may be worth 3*s.* a lb. ?

When the mixture is made and sold at 3*s.* a lb., each lb. of the cheaper tea in it brings a gain of 6*d.*, and each lb. of the dearer tea brings a loss of 9*d.* Therefore 9 lb. of the cheaper tea brings a gain of 54*d.* and 6 lb. of the dearer brings a loss of 54*d.* Hence, in order that there may be neither any gain nor any loss, for every 9 lb. of the cheaper tea we must take 6 lb. of the dearer ; therefore the proportion is 9 parts to 6, that is, *the teas must be mixed in the inverse ratio of the differences of the two prices and the mean price.*

Example 2. In what proportion should teas at 2*s.* 6*d.*, 3*s.*, 4*s.* 3*d.* and 4*s.* 9*d.*, a lb. be mixed to make a mixture worth 4*s.* a lb. ?

The first two prices are under, and the last two above, the mean price. We take equal quantities of the teas at the first two prices, and the mixture is worth 2*s.* 9*d.* a lb.; we also take equal quantities of the teas at the last two prices, and the mixture is worth 4*s.* 6*d.* a lb. Now we mix these two mixtures as in *Ex.* 1, and we find that these must be taken in the proportion of 6 to 15 or 2 to 5. Consequently the teas are mixed in the proportion of 1, 1, $\frac{5}{2}$, $\frac{5}{2}$.

Note. Instead of taking equal quantities we might take the teas in any proportion to make the first two mixtures; and consequently an example of this kind (in which the number of ingredients is more than two) may have an unlimited number of solutions.

Example 3. In what ratio must a grocer mix sugar at 6*a.* per seer with sugar at 4*a.* per seer so that by selling the mixture at 5*a.* 3*p.* per seer he may gain $\frac{1}{6}$ of his outlay?

$1\frac{1}{6}$ of the cost price of a seer of the mixture = 5*a.* 3*p.*; ∴ cost price of a seer of the mixture = 5*a.* 3*p.* ÷ $1\frac{1}{6}$ = 4*a.* 6*p.* Now proceeding as in *Ex.* 1, we find that sugar at 6*a.* per seer must be mixed with sugar at 4*a.* per seer in the ratio of (4*a.* 6*p.* − 4*a.*) to (6*a.* − 4*a.* 6*p.*), *i. e.*, of 1 to 3.

EXAMPLES. 142.

1. How must sugar at 4*a.* per seer be mixed with sugar at 5*a.* per seer to make a mixture worth 4*a.* 3*p.* per seer?

2. In what ratio must tea worth 2*s.* 7*d.* per lb. be mixed with tea worth 3*s.* 8*d.* per lb. to make a mixture worth 3*s.* per lb.?

3. Tea at 2*s.* 6*d.* per lb. is mixed with tea at 4*s.* 2*d.* per lb., and the mixture is sold for 3*s.* 5*d.* a lb.; how were they mixed?

4. In what ratio must a grocer mix coffee at 3*s.* per lb. with chicory at 7*d.* so that by selling the mixture at 2*s.* per lb. he may gain $\frac{1}{12}$ of his outlay?

5. A grocer buys black tea at 2*s.* 6*d.* per lb. and green tea at 3*s.* 9*d.* per lb.; how must he mix them so that by selling the mixture at 3*s.* per lb. he may gain $\frac{1}{6}$ of his outlay?

6. In what proportion should water and wine at 12*s.* 6*d.* a gallon be mixed to reduce the price to 10*s.* a gallon?

7. Currants at 5*d.* per lb. are mixed with currants at 9*d.* per lb. to make a mixture of 17 lb. worth 7*d.* per lb.; how many pounds of each are taken?

8. A person bought 60 md. of rice of two different sorts for ₨153. 12*a.* The better sort cost ₨3 per md. and the worse ₨2. 4*a.* per md. How many maunds were there of each sort?

9. A liquid *P* is $1\frac{3}{7}$ times as heavy as water, and water is $1\frac{2}{5}$ times as heavy as another liquid *Q*; how much of the liquid *P* must be added to 7 gallons of the liquid *Q* so that the mixture may weigh as much as an equal volume of water?

10. A mass of gold and silver weighing 9 lb. is worth £318 13*s.* 6*d.*; if the proportions of gold and silver in it were interchanged, it would be worth £129. 10*s.* 6*d.*; supposing that the price of gold is £3. 17*s.* $10\frac{1}{2}$*d.* per oz., find the proportion of gold and silver in the mass, and the price of silver per oz.

11. A merchant has wines worth 7*s*., 9*s*., 11*s*. and 15*s*. a gallon respectively: how must he mix them to obtain a mixture worth 10*s*. a gallon, using equal parts of the first two kinds, and also equal parts of the last two kinds?

12. In what proportion must a grocer mix teas at 2*s*. 6*d*., 3*s*. and 4*s*. 6*d*. per lb. to make a mixture worth 4*s*. per lb., using equal parts of the first two kinds?

13. A man has whisky worth 22*s*. a gallon, and another lot worth 18*s*. a gallon; equal quantities of these are mixed with water to obtain a mixture of 50 gallons worth 16*s*. a gallon; find how much water the mixture contains.

14. A grocer buys teas at 2*s*. 6*d*., 3*s*. and 3*s*. 9*d*. per lb. respectively: how must he mix them so as to obtain a mixture worth 3*s*. 3*d*. per lb., using the first two kinds in the proportion of 2 to 3?

15. A grocer wishes to mix teas at 2*s*., 3*s*., 3*s*. 6*d*. and 4*s*. per lb. respectively: how must he mix them (using the first two kinds in the proportion of 2 : 3, and the last two in the proportion of 3 : 4) so that by selling the mixture at 3*s*. 4*d*. per lb. $\frac{1}{15}$ of the receipts may be clear profit?

XLIII. AVERAGE VALUE.

226. The **average** or **mean value** of any number of quantities of the same kind is their sum divided by the number of them.

Example. Find the average age of four boys who are 10, 11, 13 and 14 years old respectively.

Average age $= \frac{10+11+13+14}{4}$ years $= 12$ years.

EXAMPLES. 143.

Find the average of the numbers,

1. 1, 2, 3, 4, 5.
2. 8, 10, 13, 15, 17, 20.
3. $3\frac{1}{2}, 7\frac{3}{4}, 8\frac{1}{4}, 9\frac{1}{2}, 10$.
4. 1·3, 7·6, 8·9, 3·1, ·8.

5. Find the average age of five boys who are 13, 15, 11, 9 and 8 years old respectively.

6. What was the average daily expenditure of a man in 1880, who spent R765 . 10 . 9 in the first half-year and R881 . 5 . 3 in the last?

7. The population of a town was 28750 in 1870 and 30000 in 1880; find the average annual increase between the two dates.

8. Of 20 men 12 gain £3. 7*s*. each and 8 men gain £2. 8*s*. each; what is the average gain per man?

9. Five men weighed respectively 8 st. 8 lb., 9 st. 4 lb., 10 st., 10 st. 10 lb. and 11 st. 6 lb.; what is the average weight per man?

10. If 20 chairs are bought at R5 each, and 15 at R4. 8*a.* each, and 15 more at R4 each, what is the average price of a chair?

11. A train travels 1 mile in the first 10 min., $1\frac{1}{2}$ miles in the next 10 min., 2 miles in the next, $1\frac{1}{3}$ miles in the next, and 1 mile in the next: what is the average speed of the train per hour?

12. The average weight of 6 men is 10 st.; two of them weigh 9 st. 7 lb. each; find the average weight of the others.

13. The average age of 8 men, 7 women and 1 boy is 45 years, that of the 8 men being 48 years and of the 7 women being 46; determine the age of the boy.

14. The average age of 5 children is 7 years, which is increased by 6 years when the age of the father is included; find the age of the father.

15. The average weight of 7 men is diminished by 3 lb. when one of them who weighs 10 stones is replaced by a fresh man; find the weight of the new man.

16. The average age of a class of 20 boys is 12 years; what will be the average age if 5 new boys receive admission in the class, whose average age is 7 years?

17. If the chairs in Question 10, are sold so as to gain $\frac{1}{4}$ of the cost price, what is the average selling price of a chair?

18. The average price of a chair, a table and a cot is R19; the average price of the table, the cot and a book-shelf is R22; if the price of the book-shelf be R16, find the price of the chair.

19. The average temperature for Monday, Tuesday, Wednesday and Thursday is 60°; the average for Tuesday, Wednesday, Thursday and Friday is 63°; if the ratio of the temperatures for Monday and Friday be 21 : 25, find these temperatures.

XLIV. PERCENTAGE.

227. The term **per centum** or **per cent.** means *for a hundred.*

Suppose that a trader who has a capital of R4000 gains R200; he gains R5 for every hundred of his capital. This is expressed by saying that *the trader's gain is* 5 *per cent.*

Note. The symbol % or the letters *p. c.* are used as an abbreviation for the words *per cent.*

Example 1. What fraction of a number does 5 p. c. of it denote?

5 p. c. of a number $=\frac{5}{100}$ of the number $=\frac{1}{20}$ of the number.

Example 2. How much is $6\frac{1}{4}$ p. c. of R320 ?

$$\text{The percentage} = \frac{6\frac{1}{4}}{100} \text{ of R}320 = \frac{1}{16} \text{ of R}320 = \text{R}20.$$

EXAMPLES. 144.

What fractions are denoted by the following rates per cent. ?

1. $12\frac{1}{2}$. 2. $33\frac{1}{3}$. 3. $\frac{1}{4}$. 4. $\frac{3}{8}$. 5. 125.

Find the value of

6. 5 p. c. of R700. 7. $7\frac{1}{2}$ p. c. of £140. 8. $\frac{3}{4}$ p. c. of £20.

9. 35% of 3480 men. 10. $\frac{1}{8}$% of a sq. ft. 11. 8·5% of 50 cwt.

12. A man's income is R3000 a year ; if he spends $6\frac{1}{4}$ p. c. of it each month, how much does he save in a year ?

13. Five per cent. of the total population of a town are English men ; the rest are Hindus ; if the population of the town be 37820, what is the number of Hindus ?

14. A man's income in 1871 was £500 ; in 1872 it was increased by 20 p. c. ; what was his income in 1872 ?

15. Find the difference between $\frac{3}{4}$ of R70 and $\frac{3}{4}$ p. c. of R70.

16. A testator bequeathed by will $\frac{5}{8}$ of his estate to his son, 60 p. c. of the remainder to his daughter, and the remainder to his widow ; the son got £75 more than the daughter. How much did the widow receive ?

Example 3. What *rate per cent.* does the fraction $\frac{3}{8}$ denote ?

$$\text{The fraction, } \frac{3}{8} = \frac{3 \times 100}{8 \times 100} = \frac{\frac{300}{8}}{100} = \frac{37\frac{1}{2}}{100};$$

$$\therefore \text{ rate per cent.} = 37\frac{1}{2}.$$

Example 4. What per cent. of R40 is R3 ?

$$\text{The fraction} = \frac{3}{40} = \frac{300}{40 \times 100} = \frac{\frac{300}{40}}{100} = \frac{7\frac{1}{2}}{100};$$

$$\therefore \text{ rate per cent.} = 7\frac{1}{2}.$$

EXAMPLES. 145.

What rates per cent. do the following fractions denote ?

1. $\frac{1}{4}$. 2. $\frac{1}{6}$. 3. $\frac{1}{30}$. 4. $\frac{3}{8}$. 5. $\frac{3}{7}$.

6. $\frac{7}{20}$. 7. $\frac{8}{9}$. 8. $\frac{19}{55}$. 9. $\frac{7}{16}$. 10. $\frac{22}{25}$.

What per cent. of

11. R26 is R13 ? 12. R40 is R8 ? 13. £3 is 12s. ?

14. $\cdot 25$ is $\frac{1}{7}$? 15. $\frac{1}{3}$ is $\cdot 7$? 16. $\cdot\dot{6}$ is $\cdot\dot{3}$?

17. Of 3420 men in a town, 420 died; what per cent. survived?

18. Out of a debt of ₨2500, ₨1900 is paid; what per cent. of the debt still remains unpaid?

19. The number of boys in a school in January was 320; in February it increased to 360. Find the increase per cent.

20. A mass of gunpowder is made with 2 lb. $5\frac{1}{2}$ oz. of nitre, 5 oz. of sulphur and $7\frac{1}{2}$ oz. of charcoal; find the percentage composition of the powder.

21. Standard gold contains 11 parts pure gold out of 12; what per cent. is dross?

Example 5. Of what sum of money is ₨30, 5 p. c.?

5 p. c. of the sum = ₨30,

or $\frac{5}{100}$ of the sum = ₨30;

∴ the sum = ₨30 × $\frac{100}{5}$ = ₨600.

EXAMPLES. 146.

Of what number is

1. 22, 10 p. c.? 2. 57, $4\frac{3}{4}$ p. c.? 3. 30, 120 p. c.?

4. 81, $\frac{3}{4}$ p. c.? 5. $2\frac{1}{2}$, $2\frac{1}{2}$ p. c.? 6. $3\frac{1}{2}$, $\cdot 27$ p. c.?

7. A man spends ₨3250 a year, which is $66\frac{2}{3}$ p. c. of his yearly income; find his income.

8. A man spends 60 p. c. of his income and saves ₨2000; what is his income?

9. The population of a town increased 7 p. c. from 1880 to 1883, and its population in the latter year was 13910; what was its population in 1880?

10. If a tax of 10 p. c. on the income of a man yields ₨300, how much will an income-tax of 5 pies in the ₨ produce?

MISCELLANEOUS EXAMPLES. 147.

1. The price of a bottle of red ink is 20 p. c. more than that of a bottle of black ink. If a bottle of red ink costs 12 annas, how much will a bottle of black ink cost?

2. A trader in his first year gains 8 p. c. of his capital, but in the second year loses 10 p. c. of what he had at the end of the first year, and his capital is ₨224 less than at first; find his original capital.

3. A trader's capital increased 10 p. c. every year; at the end of 3 years it was ₨6050; what was his capital at first?

4. In a mixed school 25 per cent. of the scholars are infants under 7, and the number of girls above 7 is $\frac{2}{3}$ of the boys above 7, and amounts to 36; find the number of children in the school.

5. A man spends 5 p. c. of his income in insuring life, and this part is exempted from income-tax: his income-tax which is laid at 4 pies in the rupee, amounts to ₨30. 5*a.*; find his gross income.

6. Three casks contain equal quantities of wine; a mixture is formed by taking 25 p. c. of the first cask, 35 p. c. of the second and 45 p. c. of the third; what per cent. of the whole quantity is taken?

7. Two mixed schools have 90 and 120 children respectively; in the first 60 p. c. and in the second 50 p. c. of the children are boys; what per cent. of the children in the two schools are boys?

8. In a town the numbers of male and female inhabitants are 3450 and 3020 respectively: the decrease in the former is 10 p. c., while the increase in the latter is 5 p. c. Find the increase or decrease per cent. of the total population.

9. In a mixture of coffee and chicory the coffee is 40 per cent.; to 500 lb. of the mixture a quantity of chicory is added, and then the coffee is $36\frac{4}{11}$ p. c. How many pounds of chicory are added?

10. If *A*'s income be 10 per cent. more than *B*'s, how much per cent. is *B*'s income less than *A*'s?

11. *A* sells his goods 10 per cent. cheaper than *B*, and 10 per cent. dearer than *C*; how much per cent. are *C*'s rates lower than *B*'s?

12. The price of sugar being raised 10 p. c., by how much per cent. must a man reduce his consumption of that article so as not to increase his expenditure?

XLV. COMMISSION, BROKERAGE, PREMIUM.

228. **Commission** is the sum of money paid to an agent for buying or selling goods or property of any kind. It is usually *a percentage* upon the value of goods bought or sold.

The agent is sometimes called a **broker**, especially when he buys or sells Government Promissory Notes, Shares of Companies, etc., and the commission, **brokerage.**

Premium is the sum of money paid to an *Insurance Company* which, in consideration thereof, undertakes to make good a loss incurred through fire or shipwreck, or to pay a certain sum of money after a man's death to his relatives. The instrument containing the contract is called the **Policy of Insurance**; and the stamp duty on the policy is called the **Policy duty.**

Premium is usually *a percentage* upon the sum of money which the insurer or his relatives are to receive.

Commission, Brokerage and Premium are therefore names given to a percentage in particular cases.

Example 1. An agent buys goods worth R750, and receives a commission of $2\frac{1}{2}$ per cent.; how much does he get?

$$\text{Commission} = \frac{2\frac{1}{2}}{100} \text{ of R}750 = \text{R}\frac{75}{4} = \text{R}18.\ 12a.$$

Example 2. A cargo, valued at £760, is to be insured at 5 p.c. premium; what sum must be insured that, in case of loss, the value of cargo and the premium paid may be recovered?

If every £95 (£100 − £5) be insured for £100, then in case of loss both the value of goods and premium paid will be recovered.

Now since £95 must be insured for £100,

£1 .... £$\frac{100}{95}$,

∴ £760........................ £$\frac{760 \times 100}{95}$,

or £800. *Ans.*

EXAMPLES. 148.

1. A broker purchases goods worth R5000; what is his commission at $3\frac{1}{2}$ per cent.?

2. What is the cost of insuring cargo valued at £7000, the premium being $3\frac{1}{2}$ per cent.?

3. A commission agent sells 720 bales of jute at R7 per bale; what commission does he receive at $1\frac{1}{2}$ per cent.?

4. An agent buys a house for R6750, and receives commission at R3. 12*a*. per cent.; what has his employer to pay altogether?

5. A broker received $\frac{1}{8}$ p. c. for buying Government Promissory Notes. His brokerage amounted to R35; what was the value of the Promissory Notes bought?

6. A ship is insured for $\frac{5}{4}$ of its value at $1\frac{3}{4}$ p. c. and the premium is £20; what is the ship worth?

7. The premium on a policy of insurance at 4 p. c. is R120; find the amount of the policy.

8. How much must be paid to insure a cargo worth £5720, the premium being 25*s*., policy duty 1*s*. 6*d*., and brokerage 9*s*., per £100 respectively?

9. For what sum must a merchant insure a cargo worth R9760 at $2\frac{2}{5}$ p. c. so that in case of loss both the cargo and premium may be recovered?

10. Goods worth £7740 are insured at $3\frac{1}{4}$ per cent., so that in case of loss both the value of goods and premium may be recovered; find the amount of premium paid.

11. Cargo worth £5000 is to be insured, so that in case of loss its value and all the expenses connected with its insurance may be recovered. The premium is $2\frac{1}{16}$ per cent., policy duty $\frac{13}{20}$ per cent. and brokerage $\frac{1}{4}$ per cent.; for what sum must the cargo be insured and what is the amount of the whole expense paid on insurance?

XLVI. PROFIT AND LOSS.

229. Under this head we estimate a profit or a loss, not absolutely, but in relation to the cost price, that is, as so much per cent. on the *cost price*.

Example 1. If chairs are bought at R5 each, and sold at R5. 9*a.* each, what is the gain per cent.?

The gain is 9*a.* on R5 or 80*a.*; and we have to find what per cent. of 80*a.* is 9*a.*

Now, the fraction $=\frac{9}{80}=\frac{900}{80\times 100}=\frac{\frac{900}{80}}{100}=\frac{11\frac{1}{4}}{100}$;

$\therefore$ the gain is $11\frac{1}{4}$ per cent.

Example 2. A horse is bought for R80, and is sold at a profit of 25 p. c.; what does the profit amount to, and for how much is the horse sold?

Profit $=$ 25 p. c. of R80 $=\frac{25}{100}$ of R80 $=$ R20.

$\therefore$ The horse is sold for R80 + R20, or R100.

Example 3. Some goods are bought for R90; for how much must they be sold so as to gain 10 per cent.?

The selling price $=$ 110 p. c. of cost price

$=\frac{110}{100}$ of R90 $=$ R99.

Example 4. By selling sugar at R12 per md. I gain 20 p. c.; at what price per md. did I buy it?

120 p. c. of the cost price $=$ selling price,

or $\frac{120}{100}$ of the cost price $=$ R12;

$\therefore$ the cost price $=$ R12 $\times\frac{100}{120}=$ R10.

Example 5. If 10 p. c. be lost by selling an article for R72, for how much should it have been sold so as to gain 5 per cent.?

90 p. c. of the cost price $=$ R72,

$\therefore$ 15 .... $=$ R12,

$\therefore$ 105 .... $=$ R84. *Ans.*

Example 6. By selling a house for £69 there is a loss of 8 p. c.; what would be the loss or gain per cent. by selling it for £78?

$$\text{£}69 = 92 \quad \text{p. c. of the cost price,}$$
$$\therefore \text{£}1 = \tfrac{92}{69} \quad \ldots\ldots\ldots\ldots,$$
$$\therefore \text{£}78 = \tfrac{92 \times 78}{69} \quad \ldots\ldots\ldots\ldots,$$
$$= 104 \quad \ldots\ldots\ldots\ldots$$

∴ There would be a gain of 4 per cent.

EXAMPLES. 149.

1. I sell for ₨20 that for which I gave ₨16; what is my gain per cent.?

2. At what rate per cent. is the loss on selling for £11 . 9 . $8\frac{1}{4}$ what cost £15 . 6 . 3?

3. I sell 20 articles for the same money as I paid for 25; what do I gain per cent. on my outlay?

4. If the selling price of $\frac{4}{5}$ of a number of toys be equal to the cost price of the whole, find the profit per cent.

5. 70 gallons of wine are bought for £50, and 9 gallons are lost by leakage; the remainder is sold at 1*s*. $10\frac{1}{2}$*d*. a pint; find the gain or loss per cent. on the outlay.

6. Certain articles are bought at £12. 15*s*. for 100, and are sold at $2\frac{1}{2}$ guineas for a dozen; find the gain or loss per cent.

7. A person by selling 48 yards of cloth gained the cost of 16 yards; find the gain per cent.

8. 320 maunds of rice were bought at ₨5 per maund, and sold at a loss of 5 p. c.; find the total loss and the selling price per seer.

9. A merchant buys certain goods at £6 . 19 . 3 per cwt. and pays 15*s*. per ton for expenses; at what price per lb. must he sell them so as to gain 15 p. c. on his total outlay?

10. If oranges are bought at the rate of 15 for a rupee, how many m st be sold for a rupee so as to gain 25 p. c.?

11. The cost price of a book is 7*s*. 6*d*.; if the expenses of sale be 5 p. c. upon this, and the profit 20 p. c., what would be the retail price?

12. 24 gallons of ale are bought at 2*s*. a gallon and 30 gallons of porter at 1*s*. a gallon, and they are mixed together. If 13 gallons of the mixture be lost by leakage, and 20 gallons sold at 2*s*. 3*d*. a gallon, at what price per gallon must the remainder be sold to gain 20 p. c. on the whole outlay?

13. A man having bought a quantity of tea for ₨75, sells $\frac{1}{3}$ of it at a loss of 4 p. c.; by what rate per cent. must he raise that

selling price, in order that by selling the rest at the increased price he may gain 4 p. c. on his outlay ?

14. I bought note paper at the rate of 8 annas for 5 quires, and sold it so as to gain as much on the cost of 32 quires as 8 quires were sold for ; at what price did I sell the paper per quire ?

15. A horse is sold for R440, at a loss of 12 p. c. ; how much did it cost ?

16. A quantity of sugar is sold at 6*a.* 9*p.* per seer ; the gain is $12\frac{1}{2}$ p. c. and the total gain is R15. What is the quantity of sugar sold ?

17. If oranges are sold at the rate of 11 for the rupee, and the gain is $8\frac{1}{3}$ p. c., at what rate were they purchased ?

18. A bankrupt's stock was sold for R5205 at a loss of 17 per cent. on the cost price ; had the stock been sold in the ordinary course of trade it would have realized a profit of 20 per cent. How much was it sold under the trade price ?

19. A horse was sold for R240 at a loss of $5\frac{1}{2}$ p. c. ; for what should it have been sold to gain 26 p. c. ?

20. By selling tea at 3*s.* per lb. a grocer gains only 5 p. c. ; by how much must he raise the price so as to gain 15 p. c. ?

21. If by selling 7 mangoes for R1 . 2 . $4\frac{1}{2}$ there be a profit of $16\frac{2}{3}$ per cent., at what price per dozen must they be sold to gain 20 per cent ?

22. If a man lose 4 p. c. by selling oranges at the rate of 12 a rupee, how many a rupee must he sell them so as to gain 44 p. c. ?

23. If by selling goods for R141 there be a loss of 6 p. c. ; what will be the loss or gain per cent. by selling them for R159 ?

24. Goods were sold for R37. 8*a.* with a gain of $12\frac{1}{2}$ p. c. ; what would have been gained or lost by selling them for R33. 8*a.* ?

25. Tea which cost R60 per md. is retailed at R2. 8*a.* per seer, and there is a waste of 10 p. c. ; what is the rate of profit per cent. ?

26. Sulphuric acid worth 3*d.* per lb. absorbs moisture and becomes $2\frac{1}{2}$ p. c. heavier ; what is it then worth per lb. ?

27. A merchant sells tea to a tradesman at a profit of 40 p. c., but the latter becoming bankrupt pays only 12*s.* in the £ ; how much per cent. does the merchant gain or lose on his outlay ?

28. A tradesman's prices are 30 p. c. above the cost price ; if he allows his customers 10 p. c. on his bill, what profit does he make ?

29. How much per cent. must a tradesman add on to the cost price of his goods, that he may make 20 p. c. profit after allowing his customers a reduction of 5 p. c. on his bill ?

30. The price of flour being raised 20 per cent., by how much

per cent. must a man reduce his consumption of that article so as not to increase his expenditure?

31. An article when sold at a gain of 5 p. c. yields ₨15 more than when sold at a loss of 5 p. c.; what was its prime cost?

32. A man sells an article at a loss of 10 p. c.; if he had received ₨5 more, he would have gained $12\frac{1}{2}$ p. c. What did the article cost him?

33. A piece of cloth is sold for ₨40. 10*a.* at a profit of 30 p. c. If it had been sold at ₨1. 12*a.* per yard, the profit would have been ₨12. 8*a.*; how many yards are there in the piece?

34. A man embarks his capital in three successive ventures. In the first he clears 80 p. c., and in each of the others he loses 15 p. c.; what per cent. does he gain or lose on his original outlay?

35. A boy buys a number of apples at 6 for 4*a.* and a third of the number at 4 for 2*a.*; at what rate must he sell them to gain 20 p. c. on his outlay? Supposing his total profit to be ₨4, how many did he buy?

36. How must a grocer mix teas at 3*s.* a lb. and 3*s.* 6*d.* a lb., so that by selling the mixture at 3*s.* 8*d.* a lb. he may gain 10 p. c.?

37. I must sell my stock of sugar at 3*a.* 6*p.* per lb. to gain $33\frac{1}{3}$ per cent.; by mixing it with an inferior sugar in the proportion of 4 to 1 I gain $33\frac{1}{3}$ p. c. by selling at ₨1. 9*a.* 6*p.* for $7\frac{1}{2}$ lb. Find the cost of the inferior sugar per lb.

38. A grocer proposes to sell his tea at 10 per cent. profit, but adulterates it by adding $\frac{1}{3}$ of its weight of an inferior tea which costs him $\frac{2}{3}$ of the price of the better; what profit per cent. does he make? Also in what proportion must he mix the two kinds so as to gain 20 per cent.?

39. A merchant buys 1575 cubits of cloth. He sells $\frac{1}{5}$ of it at a gain of 6 p. c., $\frac{1}{6}$ at a gain of 8 p. c., $\frac{1}{7}$ at a gain of 12 p. c. and the rest at a loss of 3 p. c. If he had sold the whole at a gain of 5 p. c. he would have received ₨120. 12*a.* more than he did What was the prime cost of a yard?

40. How must wine at 20*s.* a gallon and brandy at 45*s.* a gallon be mixed, so that by selling the mixture at 35*s.* a gallon there may be a gain of 15 p. c. on the price of the wine and 20 p. c. on the price of the brandy?

41. A mixture of two kinds of wine, at 20*s.* and 25*s.* a gallon, is sold at a gain of 10 p. c. If the two kinds had been sold separately at a gain of 15 p. c. and 8 p. c. respectively, the total profit would have been the same. In what proportion were the two kinds of wine mixed together?

42. A tradesman by means of a false balance, defrauds to the extent of 10 p. c. in buying goods, and also defrauds in selling. What per cent. does he gain on his outlay by his dishonesty?

43. A man sells a house, at a loss, for ₨400; had he sold it for ₨500 his gain would have been $\frac{2}{3}$ of his former loss; find the cost price of the house.

44. A merchant has goods worth £300; he sells one-third of them so as to lose 10 p. c. By how much per cent. should he raise that selling price in order to gain 10 p. c. on the whole?

XLVII. SIMPLE INTEREST.

230. **Interest** is money paid for the use of money lent. The money lent is called the **Principal**. The **Amount** is the sum of the principal and interest at the end of any time. The **rate of interest** is the money paid for the use of a certain sum for a certain time. Thus, if I borrow a sum of money on the condition that for the use of every rupee in the loan for a month I shall pay an interest of $\frac{1}{2}$ anna, I am said to borrow *at the rate of* $\frac{1}{2}$ anna per rupee per month. Again, if I borrow on the condition that for the use of every ₨100 in the loan for one year I shall pay an interest of ₨5, I am said to borrow *at the rate of* 5 per cent. per annum.

Note. *Per annum* means *for a year.*

231. When interest is calculated simply on the original principal it is called **Simple** Interest.

Note 1. The term interest is generally used in the sense of *simple* interest.

Example 1. Find the simple interest on ₨24 for 5 months at $\frac{1}{2}$ anna per rupee per month.

Interest on ₨1 for 1 month $= \frac{1}{2}a. = ₨\frac{1}{32}$,

$\therefore$ .... ₨24 for 1 month $= ₨\frac{1}{32} \times 24$,

$\therefore$ .... ₨24 for 5 months $= ₨\frac{1}{32} \times 24 \times 5$

$= ₨3.\ 12a.$

Hence, to find the interest we multiply the principal by 5 and by $\frac{1}{32}$, that is, we multiply it by $\frac{5}{32}$. The work in practice should stand thus:

```
         ₨.
         24
          5
    32 ) 120 ( ₨3. 12a.   Ans.
          96
          24
          16
         384 ( 12
         32
          64
          64
```

EXAMPLES. 150.

Find the simple interest on

1. ₨58 for 4 months at 6p. per rupee per month.
2. ₨76 for 9 months at 2 pice per rupee per month.
3. ₨240 for 1 year at 3p. per rupee per month.
4. ₨375 for 15 months at $\frac{3}{4}$ anna per rupee per month.
5. ₨29 for 3 years 3 months at 2p. per rupee per month.
6. ₨720 for 18 months at 4p. per rupee per month.

Example 2. Find the simple interest on ₨728 for 5 years at 4 per cent. per annum.

Interest on ₨100 for 1 year $=$ ₨4,

$\therefore$ ....₨1 for 1 year $=$ ₨$\frac{4}{100}$,

$\therefore$ ....₨728 for 1 year $=$ ₨$\frac{728\times 4}{100}$,

$\therefore$ ....₨728 for 5 years $=$ ₨$\frac{728\times 4\times 5}{100}$

$=$ ₨145. 9a. 7$\frac{1}{5}p$.

Hence we deduce the following rule :

Multiply the principal by the rate per cent. and by the number of years, and divide the product by 100.

The work should stand thus :

We divide ₨14560 by 100 by cutting off the two figures on the right ; thus the quotient is ₨145 and ₨60 is the remainder ; this remainder is equal to 960a ; this divided by 100 gives 9a. as quotient and 60a. as remainder : this remainder is equal to 720p. ; this divided by 100 gives 7·2p, as quotient.

$$\begin{array}{r} ₨. \\ 728 \\ \underline{4} \\ 2912 \\ \underline{5} \\ 100)\ ₨145,60 \\ \underline{16} \\ a.\ 9,60 \\ \underline{12} \\ p.\ 7,20 \end{array}$$

$\therefore$ Interest $=$ ₨145. 9a. 7·2p.
$=$ ₨145. 9a. 7$\frac{1}{5}p$.

Note 2. The amount may be obtained by adding the interest to the principal. Thus the amount in the above example

$=$ ₨728 $+$ ₨145. 9a. 7$\frac{1}{5}p$.

$=$ ₨873. 9a. 7$\frac{1}{5}p$.

If the amount only is wanted we may also proceed thus :

Interest on ₨100 for 5 years at 4 p. c. $=$ ₨20.

$\therefore$ The amount of ₨100 in 5 years $=$ ₨120,

$\therefore$ .... ₨1 .... $=$ ₨$\frac{120}{100}$,

$\therefore$ .... ₨728 .... $=$ ₨$\frac{728\times 120}{100}$

$=$ ₨873. 9a. 7$\frac{1}{5}p$.

EXAMPLES. 151.

N. B. The rate per cent. is understood to be *per annum* unless otherwise stated.

Find the simple interest on

1. R200 for 3 yr. at 4 p. c.
2. £300 for 4 yr. at 5 p. c.
3. R750 for 7 yr. at 6 p. c.
4. £128 for 15 yr. at 3 p. c.
5. R450 for 11 yr. at $4\frac{1}{2}$ p. c.
6. £800 for $3\frac{1}{2}$ yr. at 4 p. c.

Find the simple interest and the amount of

7. R495. 4*a.* for $2\frac{3}{4}$ yr. at 3%.
8. £325. 5*s.* for 4 yr. at $2\frac{1}{2}$%.
9. R225. 11*a.* 9*p.* for 4 years at 1 per cent. per month.

Find the amount only of

10. R250 for 2 yr. at 7 p. c.
11. £304 for 5 yr. at $4\frac{1}{2}$ p. c.
12. R335 for $3\frac{1}{2}$ years at $\frac{3}{4}$ per cent. per month.
13. £720. 8*s.* 6*d.* for $2\frac{1}{4}$ years at $2\frac{2}{3}$ per cent.
14. £329. 9*s.* $4\frac{1}{2}$*d.* for $7\frac{1}{6}$ years at $3\frac{1}{4}$ per cent.
15. £220 for 7 months at $4\frac{3}{4}$ per cent.

Note 3. When the rate per cent. and the number of years (or either of them) are fractional numbers, it is convenient first to multiply these two, and then multiply the principal by the product.

Example 3. Find the simple interest on R345. 10*a* 3*p.* for 2 years 6 months at $5\frac{1}{4}$ per cent.

Now, 2 years 6 months $=2\frac{1}{2}$ years;

and $2\frac{1}{2}\times 5\frac{1}{4}=\frac{5}{2}\times\frac{21}{4}=\frac{5\times 7\times 3}{8}$

	R.	a.	p.
	345	10	3
			5
	1728	3	3
			7
	12097	6	9
			3
8)	36292	4	3
	R45 36	8	$6\frac{3}{8}$

16

a. 5.84

12

p. 10.14$\frac{3}{8}$

See Example 2.

The interest = R45. 5*a.* 10·14$\frac{3}{8}$*p.*

= R45. 5*a.* 10$\frac{23}{160}$*p.*

EXAMPLES. 152.

N. B. When the time is given in *months* and *days*, 12 months are reckoned to the year, and 30 days to the month.

Find the simple interest on

1. ₨375 for $3\frac{1}{3}$ years at $2\frac{2}{3}$ per cent.
2. £450 for $6\frac{2}{3}$ years at $3\frac{1}{3}$ per cent.
3. £875 for 3 years 4 months 15 days at $5\frac{1}{4}$ per cent.

Find, to the nearest pie, the simple interest on

4. ₨309. 10*a*. 3*p*. for 5 months 10 days at $4\frac{1}{6}$ per cent.
5. ₨21. 15*a*. 9*p*. for 2 years 9 months at $3\frac{1}{3}$ per cent.
6. ₨101. 13*a*. for 1 year 7 months 6 days at $\frac{1}{6}$ per cent. per month.

Note 4. When interest has to be calculated from one day of the year to another, it is customary to include *one only* of the days named.

Example 4. Find the interest on £320 from January 4th to May 30th, at 3 per cent.

Number of days $=27+28+31+30+30=146$;

$$146 \text{ days} = \tfrac{146}{365} \text{ of a year} = \tfrac{2}{5} \text{ yr.}; \text{ and } 3\times\tfrac{2}{5}=\tfrac{6}{5}.$$

$$\begin{array}{rr}
 & £. \\
 & 320 \\
 & 6 \\
\hline
5\,) & 1920 \\
\hline
 & £3.84 \\
 & 20 \\
\hline
s. & 16.80 \\
 & 12 \\
\hline
d. & 9.60
\end{array}
\quad \therefore \text{ the interest} = £3.\ 16s.\ 9\tfrac{3}{5}d.$$

Note 5. It should be noted that factors of 365 are 5 and 73.

EXAMPLES. 153.

N. B. When the time is given in *days* or *years and days*, the year is taken to consist of 365 days.

Find the simple interest on

1. £400 from April 4th to June 16th at 3 p. c.
2. ₨750 from Feb. 23rd to Sep. 30th at $4\frac{1}{2}$ p. c.

3. ₨321. 8*a*. from Dec. 10th, 1887, to May 4th, 1888, at $3\frac{3}{4}$ p. c.

4. £847. 15*s*. from Jan. 1st to April 1st at $2\frac{1}{2}$ p. c.

5. ₨349. 8*a*. 9*p*. from June 1st to Oct. 4th at $5\frac{3}{4}$ p. c.

6. ₨309. 12*a*. for 1 year 73 days at $2\frac{2}{3}$ p. c.

232. Inverse questions on Simple Interest.

Example 1. At *what rate* per cent. will ₨425 amount to ₨476 in 3 years ?

Interest on ₨425 for 3 years = ₨51, (*i. e.*, ₨476 − ₨425).

∴ ....₨1 for 3 years = ₨$\frac{51}{425}$,

∴ ....₨1 for 1 year = ₨$\frac{51}{425 \times 3}$,

∴ ....₨100 for 1 year = ₨$\frac{51 \times 100}{425 \times 3}$

= ₨4 ;

∴ the rate per cent. = 4.

EXAMPLES. 154.

At what rate per cent. will

1. ₨300 amount to ₨337. 8*a*. in 5 years ?

2. ₨825 amount to ₨905. 7*a*. in 3 years ?

3. £142. 10*s*. amount to £163. 13*s*. $11\frac{1}{4}$*d*. in $4\frac{1}{2}$ years ?

4. The interest on ₨22214. 4*a*. amount to ₨462. 12*a*. 9*p*. in 7 months 10 days ?

5. A given sum of money double itself in 20 years ?

6. The interest on any sum of money be $\frac{3}{8}$ths of the amount in 20 years ?

7. The interest on £1368. 15*s*. become £14. 4*s*. $7\frac{1}{2}$*d*. from July 5th to Nov. 20th ?

8. At what rate per rupee per month will ₨250 amount to ₨312. 8*a*. in 8 months ?

Example 2. In *how many years* will £300 amount to £405 at 5 per cent. ?

Interest on £300 for 1 year = £$\frac{300 \times 5}{100}$ = £15 ; and interest on £300 for the required number of years = £405 − £300 = £105.

∴ The required number of years = $\frac{£105}{£15} = 7$.

EXAMPLES. 155.

In what time will

1. ₹475 amount to ₹532 at 4 p. c.?

2. ₹266 . 10 . 8 amount to ₹293 . 5 . 4 at 3 p. c.?

3. £1451 . 6 . 8 amount to £1667 . 4 . $4\frac{3}{8}$ at $4\frac{1}{4}$ p. c.?

4. In how many years and months will the interest on £3125 amount to £556 . 12 . $9\frac{3}{4}$ at $3\frac{3}{4}$ p. c.?

5. In how many years, months and days will ₹425 amount to ₹474 . 3 . 8 at 5 p. c.?

6. In how many days will the interest on £121 . 13 . 4 amount to £2 . 0 . 5 at $6\frac{1}{4}$ p. c.?

7. In how many years will a sum of money treble itself at $3\frac{1}{8}$ p.c.?

8. In what time will the interest on any sum of money at $6\frac{1}{4}$ p. c. be ·1875 of the principal?

9. In what time will the interest on any sum of money at 5 p. c. be $\frac{1}{8}$ of the amount?

10. On Feb. 1st, 1818, a person borrowed £400 at $6\frac{1}{4}$ p. c., promising to return it as soon as the interest amounted to £5: on what date did the loan expire?

11. In how many months will ₹3200 amount to ₹4000 at 3 pies per rupee per month?

Example 3. *What principal* will amount to ₹1000 in 10 years at $2\frac{1}{2}$ per cent.?

Interest on ₹100 for 10 years at $2\frac{1}{2}$ p. c.=₹25;

∴ ₹100 amounts to ₹125 in 10 yr. at $2\frac{1}{2}$ p. c.

Of the amount ₹125 the principal=₹100,

∴ ....₹1 ....=₹$\frac{100}{125}$,

∴ ....₹1000 ....=₹$\frac{100\times1000}{125}$

=₹800. *Ans.*

EXAMPLES. 156.

What principal will amount to

1. ₹900 in 5 years at 4 per cent.?

2. ₹4546 . 10 . 8 in $1\frac{1}{4}$ years at $5\frac{1}{4}$ per cent.?

3. £190. 15*s*. in 3 years at 4 per cent.?

4. £1153 . 9 . $4\frac{1}{2}$ in 3 years 7 months at $2\frac{3}{4}$ per cent.?

5. ₹459 . 2 . 8 in 2 years 4 months and 12 days at $6\frac{1}{4}$ per cent.?

6. R737. 8*a*. in 100 days at $3\frac{3}{4}$ per cent.?

7. R809 at $5\frac{5}{8}$ per cent. from April 20th to July 2nd?

8. R255. 7*a*. 6*p*. in $1\frac{1}{4}$ years at 3 pice per rupee per month?

What principal will produce

9. R37. 8*a*. 8*p*. interest in 4 years 3 months at $3\frac{1}{8}$ per cent.?

10. £23 . 7 . $1\frac{1}{2}$ interest in 15 years at $4\frac{5}{8}$ per cent.?

11. Find, to the nearest pie, the sum that must be invested at $3\frac{3}{4}$ per cent. for 13 years to amount to R1000.

12. Find, to the nearest penny, the principal whose interest amounts to £100 in 2 years 5 months and 10 days at 4 per cent.?

MISCELLANEOUS EXAMPLES. 157.

1. The interest on a sum of money at the end of 6 years is $\frac{3}{8}$ths of the sum itself; what rate per cent. was charged?

2. A money-lender lent a sum of money for 3 years 7 months at $1\frac{1}{2}$ pice per rupee per month. At the end of the time he received R1003 . 14 . 6: what was the sum lent?

3. A sum of money increases by $\frac{1}{12}$ of itself every year, and in 7 years it amounts to R902. 8*a*.; find the sum.

4. £275 increases by $\frac{1}{10}$ of itself per year: how long will it take to amount to £357. 10*s*.?

5. A sum of money amounts in 6 years at 5 per cent. simple interest to R442; in how many years will it amount to R510?

6. R500 is borrowed at the beginning of the year at a certain rate of interest, and after 7 months R350 more is borrowed at half the previous rate. At the end of the year the interest on both loans is R34. 6*a*. What is the rate of interest at which the first sum was borrowed?

7. What sum of money laid out at $3\frac{3}{4}$ per cent. will give R1 interest a day?

8. The principal and interest for 5 years are together R550, and the interest is $\frac{3}{8}$ of the principal: find the principal and the rate per cent. per annum.

9. The principal and interest for a certain time at $3\frac{1}{2}$ per cent. are together £450, and the interest is $\frac{2}{7}$ of the principal: find the time.

10. What sum lent out at 5 per cent. will produce in $4\frac{1}{2}$ years the same amount of interest as R500, lent out at 6 per cent., will produce in 4 years?

11. If an investment of £75 becomes £78. 15*s*. in 8 months,

what sum invested at the same rate of interest will become £201. 17*s.* 6*d.* in 10 months?

12. *A* bequeaths to *B* a certain sum of money, which after paying a legacy duty of 10 per cent. yields an income of £810 when placed at interest of 3 per cent. Find the amount bequeathed.

13. A person who pays 4*p.* in the R. income-tax, finds that a fall of interest from 4 to $3\frac{3}{4}$ per cent. diminishes his net yearly income by R47. What is his capital?

14. A sum of money doubles itself in 20 years; in how many years would it treble itself?

XLVIII. COMPOUND INTEREST.

233. When *interest*, as soon as it becomes due, is added to the principal, and interest charged upon the whole, it is called **Compound** Interest.

Example. Find the compound interest on R321. 8*a.* for 3 years at $2\frac{1}{2}$ per cent. per annum.

Now, R321. 8*a.* = R321·5, and $2\frac{1}{2}$ p. c. = 2·5 p. c.

Division by 100 is effected by moving the decimal point two places to the left.

```
          R.
        321·5
          2·5
        16075
        6430
        8·0375 = int. for 1st year.
      321·5
      329·5375 = amt. in 1 year.
           2·5
      16476875
      6590750
     8·2384375 = int. for 2nd year.
   329·5375
   337·7759375 = amt. in 2 years.
           2·5
   16888796875
   6755518750
  8·44439884375 = int. for 3rd year.
 337·7759375
 346·2203359375 = amt. in 3 years.
 321·5          = principal.
  24·7203359375 = Total Interest which
                = R24. 11a. 6·3045p.  Ans.
```

Note 1. The compound interest might also be obtained by adding together the interest for the 1st year, interest for the 2nd year and interest for the 3rd year. If the interest for $2\frac{3}{4}$ years were required, it would be obtained by adding together the interest for the 1st year, interest for the 2nd year and $\frac{3}{4}$ of the interest for the 3rd year.

Note 2. If the interest is payable *half-yearly*, the result may be obtained by finding the interest for double the number of years at half the rate per cent.

EXAMPLES. 158.

N. B. The interest is understood to be payable *yearly* unless otherwise stated.

Find, to the nearest pie, the compound interest on

1. R400 for 2 yr. at 5 p. c.
2. R520 for 2 yr. at 4 p. c.
3. R500 for $2\frac{1}{2}$ yr. at 3 p. c.
4. R1000 for 3 yr. at $4\frac{1}{2}$ p. c.

Find, to the nearest penny, the amount, at compound interest, of

5. £650 in 3 yr. at 4 p. c.
6. £320. 8*s*. in 2 yr. at $3\frac{1}{2}$ p. c.
7. £600 in $2\frac{1}{4}$ yr. at 3 p. c.
8. £250 in $2\frac{2}{3}$ yr. at $1\frac{1}{2}$ p. c.
9. Find the compound interest on R350 for 1 yr. at 4 p. c. per annum, the interest being payable half-yearly.
10. Find the compound interest on £200 for $1\frac{1}{2}$ yr. at 10 p. c. per annum, the interest being payable quarterly.

234. The following method of finding the amount at compound interest is often useful.

Example 1. Find the amount, at compound interest, of R5000 in 3 years at 4 p. c.

Amount of R100 at the end of 1 yr. = R104;

∴ .... R1 .... = R$\frac{104}{100}$;

∴ .... any sum .... = $\frac{104}{100}$ of the sum.

Also, amount of any sum at the end of 2 yr. = $\frac{104}{100}$ of the amount at the end of 1st yr.

= $\frac{104}{100}$ of $\frac{104}{100}$ of that sum

= $\left(\frac{104}{100}\right)^2$ of that sum.

Similarly, amount in 3 years = $\left(\frac{104}{100}\right)^3$ of that sum; and so on.

Hence, to find the amount of ₹5000 in 3 years, we have to nultiply ₹5000 by $(104)^3$, and divide the product by $(100)^3$.

Process :

$$\begin{array}{r}
₹\ \ 5000 \\
\underline{104\ \ } \\
520000 \\
\underline{104\ \ \ \ \ } \\
208\ \ \ \ \ \ \\
\underline{52\ \ \ \ \ \ \ \ } \\
54080000 \\
\underline{104\ \ \ \ \ \ \ } \\
21632\ \ \ \ \ \ \\
\underline{5408\ \ \ \ \ \ \ \ } \\
₹5624{\cdot}320000
\end{array}
= \text{amt. in 3 years, which}$$

$$= ₹5624.\ 5a.\ 1{\cdot}44p. \quad \textit{Ans.}$$

Division by $(100)^3$ is effected by marking off 6 decimal places in he final product.

Example 2. Find the amount of ₹400 for $2\frac{1}{2}$ years at 6 per cent ompound interest.

$$\text{Amount} = ₹400 \times \tfrac{106}{100} \times \tfrac{106}{100} \times \tfrac{103}{100} = \text{etc.}$$

Example 3. What principal will amount to ₹551. 4*a.* in 2 years t 5 per cent. compound interest ?

$$\text{Principal} \times \left(\tfrac{105}{100}\right)^2 = ₹551{\cdot}25.$$

$$\therefore\ \text{Principal} = ₹551{\cdot}25 \times \left(\tfrac{100}{105}\right)^2.$$

$$= ₹500.$$

EXAMPLES. 159.

Find, (by the method of Art. 234) to the nearest pie, the amount, at compound interest, of

1. ₹1000 in 2 yr. at 5 p. c.
2. ₹300 in 3 yr. at 3 p. c.
3. ₹700 in $2\frac{1}{2}$ yr. at 4 p. c.
4. ₹750 in 3 yr. at $4\frac{1}{2}$ p. c.
5. ₹2000 in $2\frac{1}{4}$ yr. at 4 p. c.
6. ₹4000 in $2\frac{2}{3}$ yr. at 3 p. c.
7. ₹1 in $1\frac{1}{2}$ yr. at $3\frac{1}{2}$ p. c.
8. ₹10 in $3\frac{1}{4}$ yr. at $3\frac{1}{4}$ p. c.
9. ₹3000 in $1\frac{1}{2}$ yr. at 6 p. c. per annum, interest being due nalf-yearly.
10. ₹350 in $1\frac{3}{4}$ yr. at 4 p. c. per annum, interest being due quarterly.

What sum lent at compound interest will amount to

11. £100 in 2 yr. at 5 p. c.? 12. £132. 6*s*. in 2 yr. at 5 p. c.?

13. £270. 8*s*. in 2 yr. at 4 p. c.? 14. £3413. 16*s*. in $2\frac{1}{4}$ yr. at 4 p. c.?

15. £1000 in $3\frac{1}{2}$ yr. at 6 p. c.? 16. £1 in $3\frac{3}{4}$ yr. at 8 p. c.?

MISCELLANEOUS EXAMPLES. 160.

1. Find the difference between the simple and compound interest on ₨500 for 3 years, at 4 p. c.

2. Prove that the amount at compound interest for 2 years at 2 per cent. is 1·0404 times the principal.

3. Prove that the difference between the simple and compound interest for 3 years at 5 per cent. is ·007625 times the principal.

4. The difference between the simple and compound interest on a certain sum of money for 2 years at 4 p. c. is ₨1; find the sum.

5. A person at the beginning of each year lays aside ₨1000, and employs the money at 5 p. c. compound interest; how much will he be worth at the end of 3 years?

6. The population of a town is 64000 and its annual increase is 10 per cent.; what will be the number of its inhabitants at the end of 3 years?

7. A merchant commenced with a certain capital, and gained annually at the rate of 30 per cent. At the end of 3 years he is worth ₨21970 What was his original capital?

8. A money-lender borrows money at 4 per cent. per annum, and pays the interest at the end of the year; he lends it at 6 per cent. per annum payable half-yearly, and receives the interest at the end of the year; by this means he gains ₨104. 8*a*. a year: how much money does he borrow?

XLIX. PRESENT WORTH AND DISCOUNT.

235. The **Present Worth** or **Present Value** of an amount due at the end of a given time is that sum which with its interest for the given time will be equal to the amount.

Discount is the allowance made for the payment of a sum of money before it is due.

From the definition of present worth, it follows that a debt which is due at some future period is equitably discharged by paying the present worth at once. Hence *discount is equal to the interest on the present worth*. And *Amount = Present Worth + Discount.*

Example 1. Find the present worth of ₨825, due $2\frac{1}{2}$ years hence, reckoning interest at 4 per cent.

[*N. B.* This corresponds to *Ex.* 3; Art. 232.]

₨100 amounts to ₨110 in $2\frac{1}{2}$ years at 4 p. c.

∴ Present worth of ₨110 = ₨100,

∴ .... ₨1 $= ₨\frac{100}{110}$,

∴ .... ₨825 $= ₨\frac{100 \times 825}{110}$

$= ₨750$. *Ans.*

[Discount = ₨825 − ₨750 = ₨75.]

EXAMPLES. 161.

Find the present worth of

1. ₨204, due 4 years hence, interest at 5 per cent.
2. ₨1518. 12*a.*, due in 4 years, at $5\frac{5}{8}$ per cent.
3. ₨3776. 4*a.*, due 18 months hence, at 4 per cent.
4. £1522. 1*s.* 6*d.*, due 3 years hence, at $4\frac{5}{8}$ per cent.
5. £1607. 18*s.* 4*d.*, due $4\frac{1}{2}$ years hence, at 3 per cent.
6. £1156. 2*s.* 8*d.*, due $3\frac{1}{3}$ years hence, at $4\frac{1}{2}$ per cent.
7. ₨1626, due 4 months 10 days hence, at $4\frac{1}{2}$ per cent.
8. ₨183, due 25 days hence, at 4 per cent.
9. ₨24845. 15*a.*, due 3 years hence, at $7\frac{1}{2}$ per cent. compound interest.
10. £1050. 12*s.* 6*d.*, due 2 years hence, at $2\frac{1}{2}$ per cent. compound interest.

Example 2. Find the discount on ₨600, due 4 years hence, interest being reckoned at 5 per cent.

Interest on ₨100 for 4 years at 5 p. c. = ₨20.

∴ Discount on ₨120 = ₨20,

∴ .... ₨1 $= ₨\frac{20}{120}$,

∴ .... ₨600 $= ₨\frac{20 \times 600}{120}$

$= ₨100$. *Ans.*

[Present worth = ₨600 − ₨100 = ₨500.]

EXAMPLES. 162.

Find the discount on

1. ₨355. 4*a.*, due 4 months hence, at $4\frac{1}{2}$ per cent. interest.

2. ₨2830. 3*a*. 4*p*., due 7 months hence, at 5 per cent.
3. ₨6901. 14*a*., due 9 months hence, at 3 per cent.
4. ₨2980. 6*a*. 8*p*., due 11 months hence, at 4 per cent.
5. £370. 4*s*. $8\frac{1}{4}$*d*., due 15 months hence, at $4\frac{5}{8}$ per cent.
6. £275. 6*s*. 8*d*., due $1\frac{1}{2}$ years hence, at $4\frac{1}{2}$ per cent.
7. £241. 12*s*. 4*d*., due 146 days hence, at $4\frac{1}{3}$ per cent.
8. £121. 15*s*., due 5 months hence, at $3\frac{1}{2}$ per cent.
9. ₨5208. 12*a*., due $3\frac{1}{2}$ years hence, at $4\frac{1}{2}$ per cent.
10. ₨2516. 4*a*., due 3 yr. 9 mo. 18 da. hence, at $6\frac{1}{4}$ per cent.
11. ₨6077. 8*a*. 6*p*., due 4 years hence, at 5 p. c. compound interest.
12. £413. 8*s*. 9*d*., due 2 years hence, at 5 p. c. compound interest.

236. Inverse questions.

Example 1. If the discount on ₨282. 8*a*. is ₨32. 8*a*., reckoning interest at 4 per cent., when is the amount due?

[*N. B.* This corresponds to *Ex.* 2, Art. 232.]

Amount = ₨282. 8*a*.; discount = ₨32. 8*a*.; ∴ present worth = ₨250.

∴ Interest on ₨250 for the required number of years = ₨32. 8*a*.
and interest on ₨250 for 1 year at 4 per cent. = ₨10;

∴ the required number of years $= \dfrac{\text{₨}32.\ 8a.}{\text{₨}10.} = 3\frac{1}{4}$.

∴ The amount is due $3\frac{1}{4}$ years hence.

EXAMPLES. 163.

When is the sum due, if the

1. discount on ₨1010. 10*a*. at 5 per cent. interest is ₨91. 14*a*.?
2. discount on ₨1518. 12*a*. at $5\frac{3}{8}$ p. c. is ₨268. 12*a*.?
3. discount on £520 . 17 . 6 at $4\frac{1}{2}$ p. c. is £70 . 17 . 6?
4. discount on £5747 at $3\frac{1}{2}$ p. c. is £147?
5. present worth of ₨3850 at 4 p. c. is ₨3500?
6. P. W. of ₨15941. 6*a*. 6*p*. at $3\frac{3}{4}$ p. c. is ₨13750?
7. P. W. of £8776. 6*s*. $10\frac{3}{4}$*d*. at $2\frac{1}{2}$ p. c. is £8721. 16*s*. 8*d*.?

Example 2. If the discount on ₨528. 12*a.*, due $3\frac{1}{2}$ years hence, be ₨78. 12*a.*, at what rate per cent. is the interest calculated ?

[*N. B.* This corresponds to *Ex.* 1, Art. 232.]

Amount = ₨528. 12*a.* ; discount = ₨78. 12*a.* ; ∴ present worth = ₨450.

Interest on ₨450 for $3\frac{1}{2}$ years = ₨78. 12*a.* ;

∴ .... ₨1 for $3\frac{1}{2}$ years = ₨$\frac{78\frac{3}{4}}{450}$;

∴ .... ₨1 for 1 year = ₨$\frac{78\frac{3}{4}}{450 \times 3\frac{1}{2}}$;

∴ .... ₨100 for 1 year = ₨$\frac{78\frac{3}{4} \times 100}{450 \times 3\frac{1}{2}}$ = ₨5.

∴ Rate per cent. = 5.

EXAMPLES. 164.

What is the rate of interest, if the

1. discount on ₨350, due 2 years hence, is ₨100 ?
2. discount on ₨7480, due 4 years hence, is ₨680 ?
3. discount on £397 . 2 . $2\frac{2}{5}$, due 4 years hence, is £71 . 12 . $2\frac{2}{5}$?
4. discount on £538 . 10 . $7\frac{1}{10}$, due $2\frac{3}{4}$ years hence, is £37 . 17 . $3\frac{1}{10}$?
5. present worth of ₨1260, due 4 years hence, is ₨1125 ?
6. P. W. of ₨2673. 2*a.*, due $3\frac{1}{2}$ years hence, is ₨2275 ?
7. P. W. of £2857. 10*s.*, due $12\frac{1}{4}$ years hence, is £2000 ?

237. Miscellaneous questions on P.W. and Discount.

Example 1. On what sum of money, due at the end of 2 years, does the discount, at 4 per cent., amount to ₨20 ?

Here, interest on P. W. for 2 years = ₨20.

Now, ₨8 is the interest for 2 years on ₨100,

∴ ₨4 .... ₨50,

∴ ₨20 .... ₨250 ;

∴ the P. W. = ₨250 ; and ∴ amount = ₨270. *Ans.*

Example 2. If the interest on ₨500 at 5 per cent. be equal to the discount on ₨575, when is the latter sum due ?

Here, ₨500=P. W. of ₨575 ; ∴ ₨75=interest on ₨500.
Now, the interest on ₨500 for the required number of years=₨
but the interest on ₨500 for 1 year at 5 per cent. =₨25 ;

$$\therefore \text{ the required number of years} = \frac{₨75}{₨25} = 3.$$

∴ The sum is due 3 years hence.

Example 3. The interest on a certain sum of money is ₨ and the discount on the same sum for the same time and at same rate is ₨20 ; find the sum.

Int. on the sum = Int. on P. W. + Int. on Disc.
= Disc. on the sum + Int. on Disc.

∴ Int. on the sum − Disc. on the sum = Int. on Disc.

Hence ₨2 = Int. on ₨20,

∴ ₨22 = ... ₨220. *Ans.*

Note. It should be carefully noted that *the difference betw the interest and discount on a sum of money for a certain time at a certain rate is equal to the interest on that discount for time and at that rate.*

EXAMPLES. 165.

1. On what sum of money, due at the end of 16 months, d the discount, at $4\frac{3}{4}$ per cent., amount to ₨484. 8*a.* ?

2. If the discount on a certain sum of money, due 8 mon hence, at $2\frac{1}{2}$ per cent., be ₨883 . 10 . 8, what is the sum ?

3. The discount on a certain sum of money, due at the en $2\frac{1}{4}$ years, at $2\frac{2}{3}$ per cent., is £32. 10*s.* : find the sum.

4. If the interest on ₨2275 at $3\frac{1}{2}$ per cent. be equal to discount on ₨2593. 8*a.* for the same time and at the same r when is the latter sum due ?

5. If the interest on £800 at 3 per cent. be equal to the count on £838, when is the latter sum due ?

6. If the interest on £148 for 5 years is equal to the disc at the same rate on £173. 18*s.*, due 5 years hence, what is the of interest ?

7. The interest on a certain sum of money is ₨120, and discount on the same sum for the same time and at the same is ₨100 ; find the sum.

8. The interest on a certain sum of money is ₨336, and discount for the same time and at the same rate is ₨300 : the sum.

9. The discount on a certain sum, due 2 years hence, is ₨50 and the interest on the same sum for 2 years is ₨56. 4*a.* : find the sum, and the rate per cent. per annum.

10. The interest on a certain sum, at 5 per cent., for a certain time is £50, and the discount for the same time at the same rate is £40 : find the sum, and the time.

11. If the difference between the interest and discount on a sum for 3 years at 3 per cent. be ₨1, what is the sum?

12. If the difference between the interest and discount on a certain sum of money for 9 months at 4 per cent. be 15*s.*, find the sum.

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13. *A* offers for a house ₨800, and *B* offers ₨815 to be paid at the end of 4 months. Which is now the better offer, if the rate of interest is 5 per cent. per annum?

14. A man buys 250 md. of sugar for ₨2500 payable at the end of 6 months, and the same day sells them at ₨10 per md. ready money: what does he gain by the transaction, reckoning interest at 5 per cent. per annum?

15. A tradesman marks his goods with two prices, one for ready money and the other for 6 months' credit: what ratio should the two prices bear to each other, allowing interest at 4 per cent.? If the credit price of an article be ₨50, what is the cash price?

16. Five copies of a book can be bought for a certain sum payable at the end of a year and six copies of the same book can be bought for the same sum in ready money; what is the *rate of interest*?

17. The discount on ₨550 for a certain time is ₨50; what is the discount on the same sum for twice that time?

18. The interest on £720 for a certain time is £18; find the discount on the same sum for the same time.

19. If the discount on a sum of money, due 6 months hence, at 8 p. c. be £7 . 10 . 11¼; find the P. W. of the sum.

20. A man bought an estate for £2000 and sold it immediately for £2287. 10*s.* payable at the end of 5 months. If the use of the money be reckoned at 4 per cent. per annum, what is now his gain per cent.?

21. £259. 7*s.* is due 4 years hence and £173. 18*s.*, 5 years hence: what sum at the present time is equivalent to both these sums, calculating interest at 3½ per cent.?

22. What sum must be paid now in order that a person may receive ₨2000 at the end of every year for the next 4 years, the rate of interest being 5 per cent.?

COMMERCIAL DISCOUNT.

238. A **bill** is a promise (in writing) to pay a certain sum of money at the end of a certain time.

Example. Each of the following is a bill: a **Bill of Exchange** or **Hundi** (which is a document in which one person directs another to pay to him or to some other person, a sum of money at the end of a certain time); a **Promissory Note** (which is a document in which one person promises to pay another a sum of money at the end of a certain time).

239. When a *banker* or *money-lender* purchases a bill, that is, advances money at a certain rate per cent. on the security of a bill, instead of deducting discount he usually deducts interest for the time specified adding the 3 *days of grace.* The purchaser of a bill may sell it at any time before it is due. In this case also, the second purchaser deducts interest on the amount for the time the bill has still to run adding the *three days of grace.*

Note 1. There is a *custom,* which has the force of law, by which a bill (*if not payable on demand*) always runs *three days* (called the **days of grace**) beyond the time specified. Thus a bill drawn on the 15th January, at 3 months would be **nominally** due on the 15th April, but **actually** due on the 18th. Moreover, **calendar months** are always reckoned, so that a bill drawn on the 31st January, at 3 months, would be nominally due on the 30th April and actually on the 3rd May.

Note 2. In working an example the 3 days of grace should be added *only* when the information given in the question is sufficient to enable us to determine the *exact number of days* that must elapse before the bill falls due, *and not otherwise.*

Example. A bill for £505 drawn on the 7th March at 4 months is **discounted** (*i.e., sold*) on the 28th April at 5 per cent.; how much does the holder of the bill receive, interest being deducted?

The bill is nominally due on the 7th but actually due on the 10th July; therefore the bill has still to run from 28th April to 10th July, that is, for 73 days or $\frac{1}{5}$ of a year (including *one only* of the days named).

Now, interest on £505 for $\frac{1}{5}$ yr. at 5 p. c. $= £\frac{505 \times \frac{1}{5} \times 5}{100} = £5.\ 1s.$

$\therefore$ The holder receives £505 − £5. 1*s.*, *i.e.*, £499. 19*s.*

Note 3. A banker in purchasing a bill *obtains a small advantage* by deducting interest instead of discount.

The mathematical discount is called **True Discount.**

Banker's discount (*i.e.*, interest) is called **Commercial** or **Practical** Discount.

The *banker's gain*=the difference between the *commercial* and *true* discount.

Note 4. In Arithmetic 'Discount' is always understood to mean *true* discount (and not *commercial* discount). Therefore in working examples true discount is always to be calculated unless commercial discount is expressly mentioned.

240. A *second kind of commercial discount* (which has no reference to time) is the deduction which is made by a tradesman for immediate payment of his bill. Thus when a tradesman gives notice upon his bill that he will allow 10 per cent. discount for immediate payment, he deducts ₨10 for every ₨100 in the amount of the bill. The calculation of this discount is therefore the same as of finding the simple interest on the amount of the bill for 1 year at 10 per cent.

EXAMPLES. 166.

1. Find the difference between the commercial and true discount on a bill of ₨6002. 8*a.*, due in 4 months, at $6\frac{1}{4}$ per cent.

2. A bill is drawn for £250 on June 12th at 5 months, and is discounted on Sep. 3rd at 5 per cent.; how much does the holder of the bill receive, banker's discount being allowed?

3. Find the banker's discount on a bill of £730 drawn on July 31st at 2 months and discounted on Sep. 3rd at 4 per cent.

4. What does a bill-discounter give as the present worth of a bill for ₨91. 4*a.* drawn on Sep. 4th at 5 months and discounted the same day at $6\frac{1}{4}$ per cent.?

5. A bill of ₨182. 8*a.*, nominally due on the 15th of May, is discounted on the 23rd April of the same year at 3 per cent.; what does the banker gain thereby?

6. A bill is drawn for £365 on March 31st at 3 months and discounted on June 13th at 4 per cent.; how much more was charged than the true discount?

7. The difference between the commercial and true discount on a bill for $7\frac{1}{2}$ months at 5 per cent. is ₨9; find the amount of the bill.

8. The amount of a tradesman's bill is ₨375; if he allows 10 per cent. discount, how much does he accept for immediate payment?

9. A tradesman accepts £40 for immediate payment of a bill for £50; what rate of discount does he allow?

10. If the credit price of five copies of a book is equal to the cash price of six copies of the same book, what is the *rate of discount*? [cf. Question 16, Ex. 165.]

11. A tradesman's prices are 25 p. c. above the cost price; if he allows his customers a discount of 10 p. c. on his bill, what profit does he make?

12. How much per cent. must a tradesman add on to the cost price of his goods, that he may make 20 per cent. profit after allowing his customers a discount of 10 p. c. on his bill?

L. EQUATION OF PAYMENTS.

241. When several sums are due from one person to another, payable at different times, we may be required to find the time at which they may all be paid together, so that neither the creditor nor the debtor may lose. The time so found is called the **equated time** of payment.

We give below a rule for finding the *equated time*, which will be found sufficiently accurate for all practical purposes.

Rule. Multiply each debt by the number of months [or days] after which it is due: then divide the sum of the products by the sum of the debts: the quotient will be the number of months [or days] in the equated time.

Example. If ₨400 be due from *A* to *B* at the end of 8 months, and ₨600 at the end of 10 months, when may both sums be paid in a single payment?

Number of months in the equated time $= \frac{400 \times 8 + 600 \times 10}{400 + 600} = 9\frac{1}{5}$. *Ans.*

EXAMPLES. 167.

1. ₨200 is due in 5 months and ₨400 in 8 months; find the equated time of payment.

2. ₨450 is due 2 months hence, ₨400 is due 3 months hence and ₨250 is due 4 months hence; what is the equated time?

3. Find the equated time of payment of £600, one-half of which is due in 6 months, $\frac{1}{3}$ in 9 months, and the rest in a year.

4. *A* owes *B* a debt payable in $4\frac{7}{12}$ months, but he pays $\frac{1}{4}$ in 3 months, and $\frac{1}{3}$ in 4 months: when ought the remainder to be paid?

5. *A* owes *B* on the 10th of April ₨900 due 40 days hence, he pays ₨400 on the 10th of May and ₨300 on the 20th of the same month: on what date ought he to pay the rest?

LI. STOCKS.

242. **Stock** is the name given to the money borrowed by any Government to meet national expenses, or to the *Capitals of Trading Companies*.

The money borrowed by a Government is called the **National or Public Debt**. The money lent to the Government is said to be in **Government Securities** or **Government Promissory Notes** in India, and in the **Funds** in England. A part of the National Debt in England is called the **Consolidated Annuities** or **Consols**.

When any Government raises capital by borrowing, it reserves to itself the option of paying off the principal at any future time, but promises to pay the interest at fixed periods. In India and England the interest is paid *half-yearly*.

The capital of a Trading Company is divided into **Shares**, generally of ₨100 or £100 each; those who join the company by buying one or more of these shares are called **Shareholders**. The shareholders are not required to pay the full price of their shares at once, but they have to pay it in instalments, as the business of the company progresses and **Calls** are made. The part of the capital of the company, which has thus been paid at any time, is called the **Paid-up Capital**. The profits of the company are divided periodically among the shareholders; and the moneys thus received are called **Dividends**.

When all the capital of a company has been subscribed and the company is in need of more capital, it is not usual to issue more shares like those issued at first. The company generally borrows money at a fixed rate of interest and agrees to pay the interest on this money before any dividend on the original shares is paid. Money so borrowed is called the **Preference Stock** of the company, the original capital being called the **Ordinary Stock**.

The *bonds* which are given by Joint-Stock Companies, Municipalities and similar other bodies for *borrowed capital* are called **Debentures**.

243. Stock is transferable by sale; but its price varies from a variety of causes. When the *market value* of ₨100 stock is ₨100 cash, the stock is said to be **at par**; when ₨100 stock is sold for ₨98, it is said to be **at a discount of 2 per cent.**, or, **at 2 below par**; when it is sold for ₨102, it is said to be **at a premium of 2 per cent.**, or, **at 2 above par**.

Purchases and sales of stock are usually made through *Brokers* who generally charge $\frac{1}{8}$ per cent. *on the stock bought or sold.* Thus, if the market value of ₨100 stock is ₨$97\frac{1}{2}$, the purchaser has to pay ₨$(97\frac{1}{2}+\frac{1}{8})$ and the seller receives ₨$(97\frac{1}{2}-\frac{1}{8})$.

Note. By "the 3 per cents." or "3 per cent. stock" is meant a stock, on R100 (or £100) of which is paid a dividend of R3 (or £3) per annum.

N. B. Unless the brokerage is mentioned, it need not be taken into consideration in working examples in stocks.

244. *Example* 1. What is the cost of R1500 stock in the 4 per cents. at $97\frac{7}{8}$, brokerage being $\frac{1}{8}$ per cent. ?

Cost of R100 stock $=$ R$(97\frac{7}{8}+\frac{1}{8})=$ R98,

$\therefore$ .... R1500 ...... $=$ R98×15 $=$ R1470. *Ans.*

Example 2. How much stock at $97\frac{1}{2}$ (brokerage included) can be bought for R390 ?

Amount of stock bought for R$97\frac{1}{2}=$ R100,

$\therefore$ .... R1 $=$ R$\dfrac{100}{97\frac{1}{2}}$,

$\therefore$ .... R390 $=$ R$\dfrac{100\times 390}{97\frac{1}{2}}$

$=$ R$\dfrac{100\times 390\times 2}{195}$

$=$ R400. *Ans.*

N. B. It is obvious that we have nothing to do with the rate of interest in any of the two above examples.

EXAMPLES. 168.

1. Find the cost of R2000 of 4 per cent. stock at 95.

2. Find the cost of £250 in the 3 per cent. consols at 3 below par, brokerage being $\frac{1}{8}$ p. c.

3. How much money can be obtained from the sale of R4500 stock in the Calcutta Municipal Debentures at R12 premium ? (Brokerage $\frac{1}{8}$ p. c.).

4. Find the price of the 4 per cents. when R800 stock can be purchased for R750. (B. $\frac{1}{8}$ p. c.).

5. Find the price of the $4\frac{1}{2}$ per cents. when R1700 is obtained from the sale of R1600 stock. (B. $\frac{1}{4}$ p. c.).

How much stock can be purchased by investing

6. R1350 in the 4 per cents. at R10 discount ?

7. R5062. 8*a*. in the 5 per cents. at $12\frac{3}{8}$ above par ? (B. $\frac{1}{4}$ p. c.).

8. £6909. 18*s*. in the consols at $92\frac{3}{4}$? (B. 2*s*. 6*d*. per cent.).

9. A person lays out R3750 in the purchase of 4 per cent. Govt. Securities at $93\frac{3}{8}$ and afterwards sells at $95\frac{5}{8}$; what profit does he make, the usual brokerage being charged on each transaction ?

10. A person buys £1000 3 per cent. stock at $98\frac{5}{8}$, and sells out at $96\frac{5}{8}$; how much does he lose by the transaction? (B. $\frac{1}{8}$%.)

11. A person bought Russian 5 per cent. stock at 72, and sold it when the price had risen to $75\frac{3}{4}$, thereby clearing £65; how much money did he lay out?

12. A person holds £4800 consols; if he sells out at $87\frac{5}{8}$ and invests the proceeds in the $2\frac{1}{2}$ per cents. at 81, how much of the latter stock will he hold?

13. A person invested £5330 in the 3 per cents. at 91, and when they had risen $1\frac{3}{4}$ per cent. he sold out and invested the money in the stock of the Dominion of Canada at $102\frac{1}{2}$; how much Canadian stock does he hold?

Example 3. What annual income will be derived from ₨3725 of $4\frac{1}{2}$ per cent. stock?

Income from ₨100 stock $=$ ₨$4\frac{1}{2}$,

∴ .... ₨1 .... $=$ ₨$\frac{9}{2\times 100}$,

∴ .... ₨3725 .... $=$ ₨$\frac{9\times 3725}{2\times 100}=$ ₨167. 10*a*. *Ans.*

N. B. This is merely a case of finding the interest, where the given stock is the principal.

Example 4. What annual income will be derived from ₨2042. 8*a*. invested in the 4 per cent. Govt. Securities at 102 (B. $\frac{1}{8}$%)?

Cost of ₨100 stock $=$ ₨$102\frac{1}{8}$,

∴ Income on ₨$102\frac{1}{8}$ money $=$ ₨4,

∴ .... ₨1 .... $=$ ₨$\frac{4\times 8}{817}$,

∴ .... ₨$2042\frac{1}{2}$......... $=$ ₨$\frac{4\times 8\times 4085}{817\times 2}=$ ₨80. *Ans.*

Example 5. A person transfers ₨8000 stock from 4 per cent. Govt. Securities at $98\frac{5}{8}$ to 6 per cent. Municipal Debentures at $131\frac{5}{24}$; find the alteration in his income, the usual brokerage being charged on each transaction.

Income from the 4 per cents. $=$ ₨$8000\times\frac{4}{100}=$ ₨320.

Money obtained from the sale of 4 per cents. $=$ ₨$8000\times\frac{98\frac{1}{2}}{100}$.

Income from ₨$131\frac{1}{3}$ invested in 6 per cents. $=$ ₨6,

∴ ....₨1 .... $=$ ₨$\frac{6}{131\frac{1}{3}}$,

∴ .... ₨$\frac{8000\times 98\frac{1}{2}}{100}$ .... $=$ ₨$\frac{6\times 8000\times 98\frac{1}{2}}{131\frac{1}{3}\times 100}$

$=$ ₨360.

∴ The alteration in income is ₨360 $-$ ₨320, or ₨40 *increase*.

Example 6. How much money must a person invest in the $4\frac{1}{2}$ per cent. Preference Stock of the O. E. Ry. Co. at $94\frac{1}{2}$ (brokerage included) to obtain an annual income of ₨600?

Money to be invested for ₨$4\frac{1}{2}$ income $=$ ₨$94\frac{1}{2}$,

$\therefore$ .... ₨1 .... $=$ ₨$\frac{94\frac{1}{2}}{4\frac{1}{2}}$,

$\therefore$ .... ₨600.......... $=$ ₨$\frac{94\frac{1}{2}\times 600}{4\frac{1}{2}}$

$=$ ₨12600. *Ans.*

Example 7. Find the price of 4 per cent. stock when from the investment of ₨3900 a person obtains an annual income of ₨160, brokerage being neglected.

Cost of stock producing ₨160 income $=$ ₨3900,

$\therefore$ .... ₨1 .... $=$ ₨$\frac{3900}{160}$,

$\therefore$ .... ₨4 .... $=$ ₨$\frac{3900\times 4}{160}$

$=$ ₨$97\frac{1}{2}$. *Ans.*

EXAMPLES. 169.

1. Find the half-yearly dividend on ₨3500 4 per cent. stock.

2. What annual income will be derived from ₨37250 of $4\frac{1}{2}$ per cent. stock, after paying an income-tax of 4*p*. in the ₨?

3. What amount of $3\frac{3}{4}$ per cent. stock must be bought to produce a quarterly income of £375?

4. What annual income will be derived from the investment of ₨5910 in the $4\frac{1}{2}$ per cents. at $98\frac{3}{8}$? (B. $\frac{1}{8}$%).

5. A person invests £25935 in 3 per cent. stock at 90. If the first year's dividend be invested in the same stock at 91, and the dividend for the second year at 95, what will be his income for the third year?

6. If I invest ₨16420 in the E. I. Ry. Preference Stock which pays 5 per cent. and is at $102\frac{1}{8}$, what will my clear income be, after paying an income-tax of 5*p*. in the ₨? (B. $\frac{1}{8}$ p. c.)

7. If I lay out ₨2400 in the $4\frac{1}{2}$ per cents. at 96, and after receiving the half-year's dividend sell out when they have sunk to 94, how much do I gain?

8. A person bought Bengal Bank shares at 113, and after receiving the half-year's dividend at the rate of 12 per cent. per annum sold out at $117\frac{1}{2}$, and made a profit of ₨178. 8*a*. in all; how many shares did he buy?

9. If a person invest ₨18810 in the 4 per cents. at $104\frac{1}{2}$, a

what price must he sell out after receiving the half-year's dividend to make a profit of ₹450 ?

10. A person transfers £11000 from the 4 per cents. at 92 to the 5 per cents. at 110 ; find the alteration in his income.

11. How much stock can be purchased by the transfer of ₹4000 stock from the 3 per cents. at 90 to the $3\frac{1}{2}$ per cents. at 96, and what change in annual income will be produced by the transfer ?

12. A person invested ₹5800 in the 5 per cent. Calcutta Municipal Debentures at par, and after receiving the half-yearly dividend he sells out at ₹$2\frac{1}{2}$ premium, and invests the entire proceeds in the 4 per cent. Government Securities at $95\frac{5}{32}$; what change is made thereby in his income ?

13. A person laid out ₹14500 in the $3\frac{1}{2}$ per cents. at $72\frac{1}{2}$, and when they had fallen to 68 he sold out and invested the money in the 4 per cents. at $75\frac{5}{8}$; find his gain or loss in income.

14. A person has an annual income of ₹480 from stock in the 4 per cents. ; this stock he sells out at $95\frac{7}{8}$ and invests the money in a railway stock (paying 5 p. c.) at $119\frac{9}{16}$; find the alteration in his income (B. $\frac{1}{8}$ p. c.).

15. How much money must a person invest in the 3 per cent. consols at $91\frac{3}{8}$ to obtain an annual income of £1000 ? (B. $\frac{1}{8}$ p. c.).

16. How much must a person invest in the 4 per cents. at $93\frac{3}{4}$ in order to have a clear income of ₹940 after paying an income-tax of 4*p*. in the ₹ ?

17. How much 3 per cent. stock at par must a man sell in order to purchase enough 4 per cent. stock at $114\frac{1}{58}$ to produce an income of ₹252, a brokerage of $\frac{1}{8}$ p. c. being charged on each transaction ?

18. Find the price of the 4 per cents. when the investment of ₹3750 in them produces an income of ₹160.

19. What is the price of the $4\frac{1}{2}$ per cents. when a man has an income of ₹270 by investing ₹7800 in them ? (B. $\frac{1}{8}$ p. c.)

20. A man invests £1570 in the New 4 per cent. Egyptian Annuities, and has thereupon a clear annual income of £76, after paying an income-tax of 1*s*. in the £ ; find the price of the Annuities. (B. $\frac{1}{8}$ p. c.).

Example 8. What rate of interest is obtained on money invested in the 4 per cents. at $79\frac{7}{8}$? (B. $\frac{1}{8}$ p. c.).

Interest obtained on ₹80 money = ₹4,

∴ .... ₹20 .... = ₹1,

∴ .... ₹100 .... = ₹5.

∴ Rate of interest obtained is 5 per cent.

Example 9. At what price (including brokerage) would a person have to purchase the $4\frac{1}{2}$ per cents. to get 5 per cent. for his money?

$$\text{R}5 \;=\; \text{interest on R}100 \text{ money},$$
$$\therefore \text{R}1 \;=\; \ldots\ldots\ldots\ldots\ \text{R}20 \ \ldots\ldots,$$
$$\therefore \text{R}4\tfrac{1}{2} \;=\; \ldots\ldots\ldots\ldots\ \text{R}90 \ \ldots\ldots;$$

$\therefore$ the stock must be bought at 90.

Example 10. What is the better stock to invest in, 4 per cents. at 95 or $4\frac{1}{2}$ per cents. at 105?

In the first case, interest on R95 money $=$ R4,

$\therefore$ .... R1 .... $=\text{R}\frac{4}{95}$;

in the second case, ... R105 .... $=\text{R}\frac{9}{2}$,

$\therefore$ .... R1 .... $=\text{R}\frac{9}{210}$.

It will be found that $\frac{9}{210}$ is greater than $\frac{4}{95}$; and therefore the second is the better investment.

Example 11. A person finds that if he invests his money in the 4 per cents. at 98 his income will be R42 less than if he invests it in the 5 per cents. at 112; find the sum to be invested.

In the first case, income from R1 $=\text{R}\frac{4}{98}$;

in the second case, ... R1 $=\text{R}\frac{5}{112}$;

$\therefore$ difference of income from $\text{R}1 = \text{R}\frac{5}{112} - \text{R}\frac{4}{98} = \text{R}\frac{3}{112\times7}$.

Now, $\text{R}\frac{3}{112\times7}$ = difference of income from R1,

$\therefore$ R1 $=$ .... $\text{R}\frac{112\times7}{3}$,

$\therefore$ R42 $=$ .... $\text{R}\frac{112\times7\times42}{3}$

or R10976. *Ans.*

EXAMPLES. 170.

What rate of interest is obtained by investing in the

1. 4 per cents. at 90? 2. 3 per cents. at 70? (B. $\frac{1}{3}$ p. c.).

3. A person buys £800 3 per cent. consols at 85, and £500 more when they are at 97; how much per cent. will he get for his money after deducting an income-tax of 7*d.* in the £?

4. What rate of interest do I get upon my money, if I buy Railway Shares of R75 each (which pay 4 per cent.) at 85 and pay an income-tax of 4*p.* in the R?

5. At what price would a person have to purchase the 4 per cents. to get $5\frac{1}{2}$ per cent. on his money?

6. What is the price of stock, when the $4\frac{1}{2}$ per cents. pay interest at the rate of 6 p. c. on the money invested? (B. $\frac{1}{3}$ p. c.).

7. When the 4 per cents. are at 88, what ought to be the price of the $4\frac{1}{2}$ per cents. to give the same rate of interest?

8. A man invested in the 4 per cents.; if, after deducting an income-tax of 6*p*. in the rupee, he obtained $4\frac{1}{2}$ per cent. interest on the money invested, at what price did he buy?

9. If Bank stock bought at 14 per cent. discount pay $6\frac{1}{2}$ per cent. on the investment, how much per cent. would it pay if it were bought at 28 per cent. premium?

10. Which is the better investment, 4 per cents. at 82 or 5 per cents. at 102?

11. Which is the better stock to invest in, $3\frac{1}{2}$ per cents. at $82\frac{1}{2}$ or 4 per cents. at $100\frac{1}{2}$? (B. $\frac{1}{8}$ p. c.).

12. Find the difference per cent. in income between investing in the 4 per cents. at 88 and $4\frac{1}{2}$ per cents. at 90.

13. A person finds that if he invests his money in the $4\frac{1}{2}$ per cents. at 96 his income will be greater by ₨10 than if he invests it in the 4 per cents. at 88; find the money to be invested.

14. By investing a certain sum of money in the 3 per cents. at 75 a man gets £5 . 13 . 4 less in income than he would get by investing the same sum in the $3\frac{1}{2}$ per cents. at 84; find the sum invested.

MISCELLANEOUS EXAMPLES. 171.

1. A person invested money in the 4 per cents. when they were at 95, and some more when they were at 90; find the advantage per cent. of the second purchase over the first.

2. A person invests ₨16600 in the 3 per cents. at 83, and when the funds have risen 7 per cent. he transfers $\frac{3}{4}$ of his capital to railway stock at $67\frac{1}{2}$; what dividend ought the latter to pay that he may thereby increase his income by ₨50?

3. Which is the better investment, £1256 in the $3\frac{1}{2}$ per cents. at 87, or in the railway shares at £89 per share, the dividends in the latter case being $3\frac{3}{4}$ per cent. on the sum invested?

4. A person possesses £3200 3 per cents., which he sells at $99\frac{3}{4}$; he invests the proceeds in railway shares at £56 a share, which shares pay 5 per cent. interest on £45, the amount paid on each share. By how much is his income altered by the transaction?

5. A person has ₨5000 stock in the 3 per cents. which he sells and re-invests in the $3\frac{1}{2}$ per cents. at $87\frac{1}{2}$ and increases his income by ₨5; find the price of the 3 per cents.

6. By selling £1500 3 per cents. at 95 and re-investing it I increase my income by £15 a year. If the dividend on the new shares is 8 per cent., what is the price of them?

7. What sum must be invested in the 3 per cents. at 90 to amount in $23\frac{1}{3}$ years at simple interest to £3210 cash; the price of

the stock remaining unchanged? How many years sooner would the amount be realized if the price of the stock rose to 96?

8. A gentleman in India has been receiving 12 per cent. on his capital; he goes to England, invests it in the 3 per cents. at $94\frac{3}{8}$, and his income in England is £2400 a year: what was his income in India? (£1=₨10.)

9. How much 3 per cent. stock must be sold at $87\frac{1}{2}$ to pay the present worth of ₨1645. 14*a*. due 10 months hence, at $3\frac{3}{4}$ per cent.?

10. Municipal Debentures are at 119 when the Government Securities are at $93\frac{1}{2}$, what should be their price when the Government Securities are at $71\frac{1}{2}$?

11. What is the price of the 4 per cents. when $\frac{1}{21}$ of the sum invested is received as annual interest after deducting an income-tax of 4 pies in the rupee?

12. A person invests ₨23800 partly in a $4\frac{1}{2}$ per cent. stock at $97\frac{1}{2}$ and partly in the 3 per cents. at par: if he holds twice as much 3 per cents. as $4\frac{1}{2}$, find the income that he obtains from the whole investment.

13. A man having money invested in the 3 per cents., from which he derives an income of £864, sells out at 90, and invests in shares that pay 5 per cent. interest: if his income be now increased by £336, at what price does he buy the shares?

14. What sum must have I invested in the $3\frac{1}{2}$ per cents. at 91 if, after investing £4000 more in the 3 per cents. at 75, and paying an income-tax of 7*d*. in the £ on my total gross receipts, I find my net income to be £524. 5*s*.?

15. A person who has a certain capital calculates that if he invest half his capital in the 3 per cents. at 90, and half in the 4 per cents. at par, his total income will be ₨1100; what is his capital?

16. *A* invests £3500 in buying equal amounts of 3 per cents. at $78\frac{1}{8}$ and 6 per cents. at $109\frac{3}{8}$. *B* invests the same sum, half in one stock and half in the other. Find (i) the difference in their incomes, (ii) the ratio of their rates of interest.

17. Four per cents. are at 95, and $4\frac{1}{2}$ per cents. are at 105. One person buys ₨200 stock in each, and another person invests ₨200 in each: compare the rates of interest obtained by the two on their whole investments.

18. A shareholder receives one year a dividend of 10 per cent. on his stock and pays an income-tax of 4 pies in the rupee. The next year he receives a dividend of 12 per cent. and pays an income-tax of 5 pies in the rupee. If his income is ₨394 . 5 . 4 more in the latter than in the former year, how much stock does he hold?

19. 20 shares in a company are worth ₨1600 when the dividend is at the rate of 5 per cent.; how many shares ought to be worth ₨960 when the dividend is at 6 per cent.?

20. A person invested ₨2800 in the purchase of 4 per cents. at 90 and $4\frac{1}{2}$ per cents. at 95. If his total income is ₨130, how much of each stock did he buy?

21. A man invests £1600 in the 4 per cent. stock at 80 and $7\frac{1}{2}$ per cent. stock at 125; what sums must he invest in the respective stocks to make $5\frac{3}{4}$ per cent. on his money?

22. A person, by selling 4 per cents. at 87 and investing the proceeds in the 5 per cents. at 96, finds that his income is increased by ₨17: how much 4 per cents. did he sell?

23. 4 per cent. stock, bought at $95\frac{7}{8}$, is held for 6 months at the end of which time the interest is paid; it is then sold at the same price at which it was bought: find the rate per cent. per annum of interest obtained for the money used. (Usual brokerage).

24. A person invests ₨255 in the 4 per cents. at 85, and sells part of his stock when they have risen 5 per cent. and the remainder when they have fallen 8 per cent.; he lost ₨ 11 by the transaction: how much stock did he sell out at first?

25. 5 per cent. stock is sold at 108, and with the proceeds 4 per cent. stock is bought at $91\frac{1}{8}$; after a time 4 per cent. stock is sold at $95\frac{3}{8}$ and the original stock purchased at 109, leaving a profit of ₨109 on the transaction: find the amount of 5 per cents. sold.

26. If the 3 per cents. be at 95, and the Government offer to receive tenders for a loan of £5,000,000, the lender to receive £5,000,000 stock in the 3 per cents. together with a certain sum in the $3\frac{1}{4}$ per cents., what sum in the $3\frac{1}{4}$ per cents. ought the lender to accept?

27. The present income of a railway company would justify a dividend of 6 per cent., if there were no preference shares; but as £50,000 of the stock consists of such shares which are guaranteed $7\frac{1}{2}$ per cent. per annum, the ordinary shareholders get only 5 per cent.: find the amount of the ordinary stock of the company.

28. A person buys 6 per cent. bonds, the interest on which is payable yearly and which are to be paid off at par 1 year after the time of purchase; if money be worth 5 per cent., what price should be given for the bonds?

LII. EXCHANGE.

245. Exchange means the giving or receiving a sum of money of one country equal in value to a given sum of money of another country.

The **par of exchange** between two countries denotes the *intrinsic* value of a coin of one country, as estimated in terms of a coin of the other country.

The **course of exchange** is the *actual* or *marketable* value at any time of a coin of one country, as estimated in terms of a coin of the other country.

Thus, the quantity of gold in the English sovereign being 1·261 times the quantity of gold in the French Napoleon, at par of exchange £1 is equal to 1·261 Napoleons ; but in the course of exchange £1 may be equal in value to a little more or less than 1·261 Napoleons.

Arbitration of exchange is the determination of the rate of exchange, called the **arbitrated rate,** between the first and last of a given number of places, when the rates of exchange between the first and second, the second and third, etc., of these places are known.

246. Money transactions between one country and another are usually carried on by means of **Foreign Bills of Exchange** or briefly **Foreign Bills.**

The following is the usual mode of proceeding :

Suppose I want to transmit £100 to a merchant in London. I go to a banker and buy a bill for the given amount, payable in London, at the current rate of exchange ; I then send the bill to the merchant in London, who presents it to the person on whom it is drawn and receives the amount.

247. The following table gives the principal foreign monetary systems.

France, Belgium, Switzerland	1 franc	=	100 centimes	=9½d.
Italy ...	1 lira	=	100 centesimi	=9½d.
Spain ...	1 peseta	=	100 centimos	=9½d.
Greece ...	1 drachme	=	100 lepta	=9½d.
Servia ...	1 dinar	=	100 paras	=9½d.
Bulgaria ...	1 leva	=	100 stotinkis	=9½d.
Roumania ...	1 ley	=	100 banis	=9½d.
Germany ...	1 mark	=	100 pfennige	=11¾d.
Austria ...	1 florin or gulden	=	100 kreuzers	=1s. 11¾d.
Turkey ...	1 Turkish pound	=	100 piastres	=18s. 0¾d.
Holland ...	1 florin	=	100 cents	=1s. 8d.
Portugal ...	1 milreis	=	1000 reis	=4s. 6d.
Sweden, Norway, Denmark	1 crown	=	100 ore	=1s. 0¾d.
United States	1 dollar ($)	=	100 cents	=4s. 2d.
Russia ...	1 rouble	=	100 kopecks	=R1 . 12 . 3.
China ...	1 tael = 10 mace	=	100 candareens	=R3.
Japan ...	1 yen	=	100 sen	=R2 . 7 . 6.

Note. In the countries whose names have been printed in italics in the above table, as in India, the standard coins are *silver*; in England the standard coin is *gold*; hence the value of the Rupee, etc., in English money varies with the amount of silver which can be bought for a gold sovereign. For some years past the value of silver as compared with gold has been steadily declining. A few years ago a Rupee was equal in value to about 2*s.*; now it is equal to 1*s.* 4*d.*

Example 1. Calculate the *par of exchange* between the sovereign and the rupee, supposing pure gold to be worth 15 times its weight of pure silver, having given that $46\frac{29}{40}$ sovereigns are coined from 1 lb. troy of standard gold, $\frac{11}{12}$ fine, and that a rupee contains 180 grains of silver and is $\frac{11}{12}$ fine.

The sovereign weighs $\dfrac{12\times20\times24}{46\frac{29}{40}}$ gr. or $\dfrac{12\times20\times8\times40}{623}$ gr.; and therefore it contains $\left(\frac{12\times20\times8\times40}{623}\times\frac{11}{12}\right)$ gr. or $\frac{20\times8\times40\times11}{623}$ gr. of pure gold.

The rupee weighs 180 gr.; and therefore it contains $(180\times\frac{11}{12})$ gr. or 165 gr. of pure silver, which is equivalent to $\frac{165}{15}$ gr. or 11 gr. of pure gold.

Now the number of rupees equivalent to a sovereign is the same as the number of times 11 gr. is contained in $\frac{20\times8\times40\times11}{623}$ gr.

Hence the sovereign $=\frac{20\times8\times40\times11}{623\times11}$ rupees

$=10\cdot27\ldots$ rupees.

Example 2. Find the relation between the rupee and the shilling as determined from the intrinsic value of the two coins; having given that a rupee weighs 180 grains, and is $\frac{11}{12}$ fine; and that 1 lb. troy of silver, $\frac{37}{40}$ fine, is coined into 66 shillings.

We find, as in the preceding example, that the rupee contains 165 gr. of pure silver. The shilling contains $\left(\frac{12\times20\times24}{66}\times\frac{37}{40}\right)$ gr. or $\frac{24\times37}{11}$ gr. of pure silver.

$\therefore$ 1 rupee $=(165\div\frac{24\times37}{11})$ shillings

$=2\cdot043\ldots$ shillings.

Example 3. Exchange R550 for English money at 1*s.* 8*d.* per rupee.

R1 $=1s.\ 8d.$,

$\therefore$ R550 $=1s.\ 8d.\times550$

$=£45.\ 16s.\ 8d.$ *Ans.*

Example 4. Determine the *course of exchange* between India and England, when Indian money is at a discount of 25 p. c., having given that at par 1 rupee = 2 shillings.

[Indian money being at a discount of 25 p. c. means that it is worth 25 p. c. less English money than it would be if it were at par.]

$$\text{At par} \quad \text{R}1 = 2s,$$
$$\therefore \text{ at 25 p. c. disc.} \quad \text{R}1 = 2s. - \tfrac{1}{4} \text{ of } 2s.$$
$$= 1s.\ 6d.$$

$\therefore$ The course of exchange is 1*s*. 6*d*. per R1.

Example 5. If the rate of exchange between Calcutta and London is at 1*s*. 9*d*. per rupee, and that between London and Paris is at 25 francs per £1, what is the *arbitrated rate of exchange* between Calcutta and Paris?

$\text{R}1 = 1s.\ 9d. = \pounds\frac{7}{80} = \frac{7}{80} \times 25$ francs $= 2\frac{3}{16}$ francs. (See Art. 205).

$\therefore$ The required rate is $2\frac{3}{16}$ francs per rupee.

EXAMPLES. 172.

1. Convert R3782 to English money, the course of exchange being 1*s*. $5\frac{1}{2}$*d*. per R.

2. Exchange £329. 7*s*. 6*d*. for Indian money at R11. 4*a*. per £.

3. A Spanish pistole is worth 15*s*. and an Austrian ducat 9*s*. 5*d*.; how many ducats are equivalent to 226 pistoles?

4. A French Napoleon or 20-franc piece is worth £·79; find, to the nearest farthing, the value in English money of 123·21 francs.

5. A bill bought in Calcutta at 1*s*. 6*d*. a rupee, is sold in New York at 4*s*. 3*d*. a dollar; determine the course of exchange between New York and Calcutta.

6. If £3 = 20 thalers; 25 thalers = 93 francs; 27 francs = 5 scudi; and 62 scudi = 135 gulden; how many gulden can I get in exchange for £11?

7. Find the arbitrated rate of exchange between Vienna and Calcutta in rupees for 1 florin, when the exchange between Calcutta and London is R3 for 5*s*., between London and Paris is 25 francs for £1, between Paris and Berlin 5 francs for 4 marks, and between Berlin and Vienna 2 marks for 1 florin.

8. If a thaler is equivalent to 40 kreuzers, 10 silber-groschen and half a gulden, and if 30 silber-groschen make a thaler and 60 kreuzers make a gulden, how many gulden are worth 8 thalers?

9. If R1 in England exchanges for 1*s*. $5\frac{3}{8}$*d*., and if £1 in India exchanges for R13. 5*a*. 6*p*., how much do you lose in R960 by the two exchanges?

10. A person in Calcutta wishes to remit a debt of 240 dollars to New York when the exchanges are 1 dollar = R2. 13*a*., R1 = 1*s*. 6*d*. and 25*s*. = 6 dollars. Is it more advantageous for him to remit directly to New York or circuitously through London?

11. A merchant in London is indebted to one at St. Petersburg 15000 roubles: the exchange between St. Petersburg and London is 50*d.* per rouble, between St. Petersburg and Amsterdam 91*d.* Fl. per rouble, and between Amsterdam and London 36*s.* 3*d.* Fl. per £ sterling. What difference will it make if the London merchant is drawn upon through Amsterdam or direct?

12. If in London I get £1 for 25 francs 20 centimes, what shall I gain or lose per cent. by taking French money into Bavaria when the exchange is 11 gulden 40 kreuzers for £1, and 8 gulden 20 kreuzers for a Napoleon? (1 Napo. = 20 fr.; 1 fr. = 100 centimes; 1 guld. = 60 kreuz.)

13. The Indian bazar maund is equal to $82\frac{2}{7}$ lb. avoir., and the rupee is equal to 2*s.* If 1 md. of wheat cost ₨3, what will be the price in English money of 1 cwt.?

14. Exchange 380 dollars for English money when it is at a discount of 5 per cent., given that at par 1 dollar = 4*s.* 2*d.*

15. Exchange ₨660 for English money when it is at a premium of 10 per cent., it being given that at par ₨1 = 1*s.* $10\frac{1}{2}$*d.*

16. If India exchanges with England at a loss of 15 per cent. when the course of exchange is 1*s.* 5*d.* per ₨, what is the par of exchange?

17. A merchant in Calcutta wishes to remit to London ₨900, a rupee being equal to 2*s.*; for what sum in English money must he draw his bill when bills *on* London are at a premium of $12\frac{1}{2}$ per cent.?

18. I pay ₨51000 to a bank for a bill of exchange payable in London. The rate of exchange is 1*s.* $10\frac{1}{2}$*d.* for the rupee, and the bank charges me 2 per cent on the amount payable in England. How much will my agent in London receive?

19. A person in London owes another in St. Petersburg 460 roubles, which must be remitted through Paris. He pays the requisite sum to his broker when the exchange between London and Paris is 23 francs for £1, and between Paris and St. Petersburg 2 francs for one rouble. The remittance is delayed until the rates of exchange are 24 francs for £1, and 3 francs for 2 roubles. What does the broker gain or lose by the transaction?

20. The exchange of Calcutta on London at 3 months is 1*s.* $4\frac{1}{8}$*d.* per ₨; find the exchange at sight, reckoning 5 per cent. per annum.

21. Calculate the par of exchange between the gold mohur, weighing 180 grains, $\frac{11}{12}$ fine, and the U. S. eagle, weighing 258 grains, $\frac{9}{10}$ fine.

22. Calculate the par of exchange between the Napoleon and the rupee, supposing pure gold to be worth 15 times its weight of pure silver; being given that $16197\frac{1}{2}$ grains of French standard

gold, $\frac{9}{10}$ fine, is coined into 155 Napoleons, and that a rupee contains 180 grains of silver, $\frac{11}{12}$ fine.

23. From 3465 grains of fine silver are coined 14 thalers; find the value of a thaler, when a pound troy of Indian standard silver, of which 11 parts out of 12 are fine, is worth R32.

24. If 1 lb. of English standard silver, of which 37 parts in 40 are pure silver be worth 62*s.*, find the value of a Hyderabad rupee which weighs 7 dwt. 17 gr., and has a fineness of 30 parts in 31.

25. The gold coinage of one nation contains 1 part of silver to 11 parts of gold; that of another nation, 1 part of silver to 23 parts of gold. It is found that 59 of the first weigh as much as 123 of the second. The intrinsic value of silver is one-sixteenth that of gold. Determine the par of exchange.

LIII. METRIC SYSTEM AND DECIMAL COINAGE.

248. The **Metric System** of weights and measures, which originated in France, has been introduced to a greater or less extent into almost all the countries of Europe. It is also nearly always used in scientific treatises.

The Tables of weights and measures in the metric system are constructed upon one uniform principle, by attaching the following prefixes to each of the *units.*

GREEK PREFIXES.				LATIN PREFIXES.			
Deca	means	10	times.	**Deci**	means	10th	part of.
Hecto	,,	100	,,	**Centi**	,,	100th	,,
Kilo	,,	1000	,,	**Milli**	,,	1000th	,,
Myria	,,	10000	,,				

In this system the fundamental unit of length is the **metre**, whence the system is called the *metric* system. The metre is equal to 39·37079... inches, and was originally taken to be the ten-millionth part of a quarter of the terrestrial meridian. An error has however been since found in the measurement of the terrestrial meridian, and the metre therefore is not exactly the length it was stated to be.

TABLE.

10 millimetres (mm.)	=	1 centimetre (cm.).
10 centimetres	=	1 decimetre (dm.).
10 decimetres	=	1 metre (m.).
10 metres	=	1 decametre (Dm.).
10 decametres	=	1 hectometre (Hm.).
10 hectometres	=	1 kilometre (Km.).
10 kilometres	=	1 myriametre (Mm.).

1 metre = about $1\frac{1}{11}$ yards; 1 kilometre = about 5 furlongs.

Example. 23564 m. 7 dm. 9 cm. 8 mm.=23564798 mm. =2356479·8 cm.=235647·98 dm.=23564·798 m.=2356·4798 Dm. =235·64798 Hm.=23·564798 Km. =2·3564798 Mm. =2 Mm. 3 Km. 5 Hm. 6 Dm. 4·798 m.

The unit of area is the **square metre.** In measuring land the unit used is a *square decametre,* called an **are,** and the only multiple and submultiple used are the *hectare* (=100 ares=a square hectometre) and the *centiare* (=$\frac{1}{100}$ of an are=a square metre).

TABLE.

100 sq. millimetres (mmq.)	=	1 sq. centimetre (cmq.).
100 sq. centimetres	=	1 sq. decimetre (dmq.).
100 sq. decimetres	=	1 sq. metre (mq.).
100 sq. metres	=	1 sq. decametre (Dmq.).
100 sq. decametres	=	1 sq. hectometre (Hmq.).
100 sq. hectometres	=	1 sq. kilometre (Kmq.).
100 sq. kilometres	=	1 sq. myriametre (Mmq.).

1 centiare (ca.)	=	1 sq. metre.
100 centiares	=	1 are (a.) [=1 sq. decametre].
100 ares	=	1 hectare (ha.) [=1 sq. hectometre].

1 are=1076·43 sq. feet; 1 hectare=about $2\frac{1}{2}$ acres.

Example 1. 2 Dmq. 64 mq. 9 dmq. 34 cmq.=2640934 cmq. =26409·34 dmq.=264·0934 mq.=2·640934 Dmq.=·02640934 Hmq. =·0002640934 Kmq.

Example 2. 73204 ca.=732·04 a.=7·3204 ha.=7 ha. 32 a. 4 ca.

The unit of volume is the **cubic metre.** The multiples of the cubic metre are seldom used. In measuring wood the cubic metre is called a **stere,** and 10 steres make a *decastere.*

TABLE.

1000 cu. millimetres	=	1 cu. centimetre.
1000 cu. centimetres	=	1 cu. decimetre.
1000 cu. decimetres	=	1 cu. metre.

1 cu. metre=1 stere; 10 steres=1 decastere.
1 cu. metre or stere=35·317 cu. feet (nearly).

Example. 27·03567 cu. m.=27035·67 cu. dm.=27035670 cu. cm.=27 cu. m. 35 cu. dm. 670 cu. cm.

The unit of capacity, both for liquids and dry goods, is the **litre**, and is equal to a *cubic decimetre.*

TABLE.

10 millilitres (ml.)	=	1 centilitre (cl.).
10 centilitres	=	1 decilitre (dl.).
10 decilitres	=	1 litre (lit.).
10 litres	=	1 decalitre (Dl.).
10 decalitres	=	1 hectolitre (Hl.).
10 hectolitres	=	1 kilolitre (Kl.).

Since 1 litre = 1 cubic decimetre, 1000 litres = 1 kilolitre, and 1000 cubic decimetres = 1 cubic metre, ∴ 1 kilolitre = 1 cubic metre.

1 litre = ·035317 cu. feet = $1\frac{3}{4}$ pints nearly; 1 kilolitre = 35·317 cu. feet (nearly).

Example. 3025·407 lit. = 30254·07 dl. = 302540·7 cl. = 3025407 ml. = 302·5407 Dl. = 30·25407 Hl. = 3·025407 Kl. = 3 Kl. 2 Dl. 5 lit. 4 dl. 7 ml.

The unit of weight is the **gram** which is the weight of a *cubic centimetre* of distilled water at its maximum density.

TABLE.

10 milligrams (mg.)	=	1 centigram (cg.).
10 centigrams	=	1 decigram (dg.).
10 decigrams	=	1 gram (gr.).
10 grams	=	1 decagram (Dg.).
10 decagrams	=	1 hectogram (Hg.).
10 hectograms	=	1 kilogram (Kg. or Kilo.).
10 kilograms	=	1 myriagram (Mg.).

Since 1 litre = 1000 cubic centimetres, and 1 kilogram = 1000 grams, ∴ the weight of a litre of water = 1 kilogram. The weight of a kilolitre (1 cubic metre) of water is 1000 kilograms and is called a *tonneau de mer* or *millier*. A *quintal* = 100 kilograms.

1 gram = 15·4323487 grains; 1 kilogram = $2\frac{1}{5}$ lb avoir. nearly.

Note. Act XXXI of 1871 of the Government of India enacts that the unit of weight shall be the **Ser** equal in weight to the French Kilogram, and the unit of capacity shall be the measure which holds one such Ser of water at its maximum density weighed *in vacuo*. These units, however, have not yet been practically adopted.

French Money.

10 centimes (c.)	=	1 decime.
10 decimes	=	1 franc (fr.).

Accounts are kept in francs and centimes only; thus "32·78 francs" is read 32 francs 78 centimes.

The *Franc* is a silver coin composed of 9 parts of silver and 1 part of copper, and weighs 5 grams. It is equal to $9\frac{3}{5}d.$ nearly. The *Napoleon* is a gold coin = 20 francs.

The Proposed Decimal Coinage of Great Britain.

10 mils (m.)	=	1 cent. (c.).
10 cents	=	1 florin. (f.).
10 florins	=	£1.

249. The great advantage of a decimal system of weights and measures is, as we have seen, that a compound quantity can be reduced to a simple quantity, and *vice versa*, without going through the processes of multiplication and division. Hence compound rules are replaced by the corresponding simple rules.

Example 1. Express 7 hectares 34 ares 6 centiares as a decimal of a sq. kilometre.

7 ha. 34 a. 6 ca. = 73406 ca. = 73406 sq. metres = 734·06 sq. decametres
= 7·3406 sq. hectometres = ·073406 sq. kilometres.

Example 2. A wheel makes 1230 revolutions in passing over 2 kilometres 5 hectometres 9 metres 2 decimetres, what is its circumference?

2 Km. 5 Hm. 9 m. 2 dm. = 2509.2 m.; 2509·2 ÷ 1230 = 2·04; ∴ the circumference reqd. = 2·04 metres = 2 metres 4 centimetres.

Example 3. A cubic foot of alcohol weighs 94 lb.; find the weight of a litre in grams, supposing a litre to be equal to ·035 cu. ft., and a gram 15·43 grains.

Weight of a litre of alcohol = ·035 × 94 lb.

$$= \cdot035 \times 94 \times 7000 \text{ grains}$$

$$= \frac{\cdot035 \times 94 \times 7000}{15\cdot43} \text{ grams}$$

$$= 1492\cdot5\ldots\text{grams.}$$

Example 4. Cloth is sold at 21 fr. 80 c. per metre; what is the corresponding price per yard in English money, if £1 be worth 25 fr. 25 c.? [1 metre = 39·37 inches.]

$$1 \text{ yard} = 36 \text{ inches} = \frac{36}{39{\cdot}37} \text{ metres};$$

$$\therefore \text{ cost of } 1 \text{ yard} = \frac{36 \times 2180}{39{\cdot}37} \text{ centimes} = £\frac{36 \times 2180}{39{\cdot}37 \times 2525}$$

$$= 15s.\ 9\tfrac{1}{2}d. \text{ nearly.}$$

Example 5. Add together £3. 7f. 2c. 3m., £9. 2f. 0c. 4m., and 7f. 3c.

$$\begin{array}{r} \text{mils} \\ 3723 \\ 9204 \\ \underline{730} \\ 13657 \end{array} \text{ mils} = £13.\ 6\text{f}.\ 5\text{c}.\ 7\text{m}. \quad \textit{Ans.}$$

Example 6. Multiply 7f. 9c. 3m. by 32.

$$\begin{array}{r} \text{mils} \\ 793 \\ \underline{32} \\ 1586 \\ \underline{2379} \\ 25376 \end{array} \text{ mils} = £25.\ 3\text{f}.\ 7\text{c}.\ 6\text{m}. \quad \textit{Ans.}$$

250. We can easily *decimalise* a sum expressed in £. *s.* *d.* and change decimal coinage into £. *s.* *d.*

Example 1. Express £7. 15*s.* 7½*d.* in decimal coinage.

$$\begin{array}{r|l} 4 & 2{\cdot}0 \\ 12 & 7{\cdot}5 \\ 20 & 15{\cdot}625 \\ & £7{\cdot}78125 = £7.\ 7\text{f}.\ 8\text{c}.\ 1{\cdot}25\text{m}. \quad \textit{Ans.} \end{array}$$

Example 2. Express £9. 3f. 9c. 8m. in £. *s.* *d.*

$$\begin{array}{r} £9{\cdot}398 \\ \underline{20} \\ s.\ 7{\cdot}960 \\ \underline{12} \\ d.\ 11{\cdot}520 \end{array}$$

$\therefore$ £9. 3f. 9c. 8m. = £9. 7*s.* 11·52*d.*

EXAMPLES. 172a.

Reduce

1. 2305000 millimetres to kilometres.
2. 304007 centimetres to kilometres, etc.
3. 1203270 millimetres to decametres, etc.
4. 75 kilometres 7 decametres 3·05 metres to millimetres.
5. 3070·5086 decametres to kilometres, etc.
6. 23 sq. kilometres 8 sq. decametres 7 sq. metres to sq. metres.
7. 50 sq. kilometres 6 sq. hectometres 4 sq. metres to sq. decametres.
8. 40740 centiares to hectares, etc.
9. 8 hectares 7 ares to centiares.
10. 36·307 sq. hectometres to hectares, etc.
11. 3012035 cu. centimetres to cu. metres, etc.
12. 5 cu. metres 27 cu. decimetres 4 cu. centimetres to cu. millimetres.
13. 40700302 millilitres to kilolitres, etc.
14. 3040600 centigrams to myriagrams, etc.
15. 1375 centimes to francs, etc.

16. A man walks 7·92 kilometres in 2 hours; how many metres does he walk in a second?

17. The circumference of a bicycle wheel is 4 metres 8 centimetres; how many times will it revolve in going 16·83 kilometres?

18. If 25 horses eat 676 kilo. 575 gr. of corn in 9 days, how long will 240 kilo. 560 gr. serve 16 horses?

19. The weight of 226 equal parcels is 1 tonneau 921 kilograms; find the weight of each.

20. If 27 decalitres 8 centilitres of wine cost 67 francs 52 centimes, find the cost of 15 litres.

21. An estate containing 30 hectares 50 ares is divided into 1000 fields of equal area; find the area of each.

22. How much wheat at 19 francs 55 centimes per hectolitre ought to be given in exchange for 312 hectolitres 80 litres of barley at 1 franc 25 centimes per decalitre?

23. Express a yard in terms of the metre, supposing a metre to be equal to 39·37 inches.

24. Express a kilometre as a decimal of a mile, if a metre be 39·37 inches.

25. The standard height of the barometer is 760 mm. Find this height in inches. [1 metre=39·3708 inches.]

26. Express a pound avoir. in grams, a gram being equal to 15·43 grains.

27. If a cubic inch of air weigh ·31 grains, what will be the weight in grams of a litre of air, having given that a cubic metre is equal to 35·3 cubic feet, and a gram 15·43 grains.

28. A gallon of water weighs 10 lb ; find its volume in cubic centimetres, supposing a kilogram to be equal to $2\frac{1}{5}$ lb.

29. Mahogany is 55 lb. to the cubic foot ; find the weight of a decastere of Mahogany in tonneaux and kilograms, supposing a cubic metre to be 35·3 cubic feet, and a kilogram $2\frac{1}{5}$ lb.

30. An inch is 2·54 centimetres, and a kilogram is 2·2 lb., find the pressure of the atmosphere in grams per sq. centimetre, supposing it to be 15 lb. avoir. to the square inch.

31. If a kilolitre be 220 gallons, find the value, in English money, of a pint of liquid which is worth 33 francs the decilitre, 1200 francs being equal to £47.

32. A decimetre is equal to 3·937 inches, and a cubic inch of water weighs 252·45 grains. Express a kilogram in pounds avoir. correct to two decimal places.

33. A gallon contains 277·274 cubic inches, a cubic decimetre is 61 cubic inches, and a kilogram is $2\frac{1}{5}$ lb. ; calculate the weight in pounds of a gallon of water.

LIV. INVOICES AND ACCOUNTS.

251. (i) *Specimen of an Invoice.*

Calcutta April 23, 1889.

Charles Smith, Esq.,

Bought of William Moran & Co.,

7, Bankshall Street.

	R.	a.	p.
8 yd. of flannel at R1. 4*a*. per yd.	10	0	0
10 yd. of calico at 3*a*. 6*p*. per yd.	2	3	0
2 pairs of gloves at R1. 9*a*. 9*p*. per pair	3	3	6
R	15	6	6

(ii) *Specimen of an Account.*

Calcutta, June 30, 1889.

Charles Smith, Esq.,

To William Moran & Co.,

7, Bankshall Street.

1889.			R.	a.	p.
April 23,	To goods, as per Invoice	... ...	15	6	6
May 7,	To ditto	... ...	3	7	3
„ 13,	To ditto	... ...	9	0	0
June 12,	To ditto	... ...		7	6
		R	28	5	3

(iii) *Specimen of a Detailed Account.*

Calcutta, June 30, 1889.

Charles Smith, Esq.,

To William Moran & Co.,

7, Bankshall Street.

1889.			R.	a.	p.
April 23,	8 yd. of flannel at R1. 4*a*. per yd....	...	10	0	0
„	10 yd. of calico at 3*a*. 6*p*. per yd. ...	...	2	3	0
„	2 pairs of gloves at R1. 9*a*. 9*p*. per pair...		3	3	6
May 7,	3 dozen stockings at R6 per doz....	...	18	0	0
May 13,	13 yd. of linen at 8*a*. 6*p*. per yd. ...	...	6	14	6
June 12,	20 yd. of carpet at R3. 8*a*. per yd. ...	...	70	0	0
„	4 pairs of socks at R1 per pair ...	...	4	0	0
		R	114	5	0

Note. Invoices and Accounts are called **Bills.** Each separate entry in a bill is called an **item.** When an account is sent to a buyer it is said to be **rendered.**

LV. PROBLEMS IN HIGHER ARITHMETIC.

252. *Example* 1. A person has a number of oranges to dispose of; he sells half of what he has and 2 more to A, $\frac{1}{3}$ of the remainder and 4 more to B, $\frac{1}{4}$ of the remainder and 6 more to C; by which time he has disposed of all he had. How many had he at first?

When he had given $\frac{1}{4}$ of his oranges to C he had 6 left; therefore this is $(1-\frac{1}{4})$ or $\frac{3}{4}$ of the number he had before C came, and therefore he had $6\times\frac{4}{3}$ or 8 before C came; therefore he had $(8+4)$ or 12 before he had given 4 oranges to B; but this is the number

he had left when he had given $\frac{1}{3}$ of his oranges to B; therefore this is $(1-\frac{1}{3})$ or $\frac{2}{3}$ of the number he had before B came, and therefore he had $12 \times \frac{3}{2}$ or 18 before B came; therefore he had $(18+2)$ or 20 before he had given 2 oranges to A; but this is the number he had left when he had given $\frac{1}{2}$ of his oranges to A; therefore he had 20×2 or 40 before A came: that is, he had 40 oranges at first.

Example 2. The expenses of a family when rice is at 12 seers for a rupee are ₨80 a month; when rice is at 15 seers for a rupee the expenses are ₨77 a month; what will they be when rice is at 18 seers for a rupee?

The prices of a seer of rice in the three cases are ₨$\frac{1}{12}$, ₨$\frac{1}{15}$ and ₨$\frac{1}{18}$ respectively; ∴ the price of a seer is first reduced by ₨$(\frac{1}{12}-\frac{1}{15})$ or ₨$\frac{1}{60}$, and finally by ₨$(\frac{1}{12}-\frac{1}{18})$ or ₨$\frac{1}{36}$. Now, when the saving on a seer of rice is ₨$\frac{1}{60}$ the total saving is ₨$(80-77)$ or ₨3; ∴ when the saving on a seer is ₨$\frac{1}{36}$ the total saving will be ₨$\frac{3 \times 60}{36}$ or ₨5. ∴ The reqd. expenses = ₨$(80-5)$ = ₨75.

Or thus: When the saving on each seer of rice is ₨$\frac{1}{60}$ the total saving is ₨3; ∴ the number of seers of rice required by the family per month = ₨3 ÷ ₨$\frac{1}{60}$ = 180; and the price of 180 seers at 12 seers for a rupee is ₨15; ∴ the other expenses of the family = ₨$(80-15)$ = ₨65. Again, the price of 180 seers at 18 seers for a rupee is ₨10; ∴ the total expenses when rice is at 18 seers for a rupee will be ₨$(65+10)$ or ₨75.

Example 3. A labourer was engaged for 36 days, on the agreement that for every day he worked he should have 4*a.*, but that for every day he absented himself he would be fined 2*a.* He received ₨7. 8*a.* at the end of the time; how many days was he absent?

If he had worked all the 36 days he would have received ₨9; ∴ through absence he lost (₨9 − ₨7. 8*a.*) or ₨1. 8*a.* But for each day of absence he actually loses (4*a.* + 2*a.*) or 6*a.*; ∴ the number of days he was absent = ₨1. 8*a.* ÷ 6*a.* = 4.

Example 4. I have to be at a certain place in a certain time, and I find that if I walk at the rate of 4 miles per hour I shall be five minutes too late, and if at the rate of 5 miles per hour I shall be 10 minutes too soon; what distance have I to go?

If I walk 4 miles an hour I require 15 minutes more time in going the distance than if I walk 5 miles an hour. And in walking one mile I require 3 minutes more at the former rate than at the latter. Hence I have to go a distance of 5 (*i.e.*, $15 \div 3$) miles.

Example 5. I have a certain sum of money to be distributed among a certain number of boys, and I find that if I give ₨3 to each I shall spend ₨4 too little, but that if I give ₨5 to each I shall spend ₨6 too much. How much have I to spend?

If I give ₨5 instead of ₨3 to each I require ₨2 more per head and (₨4+₨6) or ₨10 more on the whole; ∴ the number of boys=₨10÷₨2=5; and ∴ I have to spend (₨3×5+₨4) or ₨19.

Example 6. A lb. of tea and 4 lb. of sugar cost 5*s.*; but, if sugar were to rise 50 per cent. and tea 10 per cent., they would cost 6*s.* 2*d.*: find the cost of the tea and the sugar per lb.

If both tea and sugar were to rise 50 p. c., the cost of 1 lb. of tea and 4 lb. of sugar would be 7*s.* 6*d.*; but tea rises only 10 p. c., ∴ 40 p. c. of the cost of a lb. of tea=7*s.* 6*d.*−6*s.* 2*d.*=1*s.* 4*d.*; ∴ the cost of a lb. of tea=3*s.* 4*d.*; ∴ the cost of 4 lb. of sugar =5*s.*−3*s.* 4*d.*=1*s.* 8*d.*; and ∴ 1 lb. of sugar costs 5*d.*

Example 7. Three tramps meet together for a meal; the first has 3 loaves, the second 2, and the third, who has his share of the bread, pays the other two 5*d.*; how ought they to divide the money?

Each eats $\frac{5}{3}$ loaves; ∴ the first has given $(3-\frac{5}{3})$ loaves and the second $(2-\frac{5}{3})$ loaves to the third: ∴ the 5*d.* given by the third ought to be divided in the ratio of $(3-\frac{5}{3})$ to $(2-\frac{5}{3})$, *i.e.*, of 4 to 1; ∴ the first will take 4*d.* and the second 1*d.*

Example 8. The sum of the ages of *A* and *B* is now 45 years, and their ages 5 years ago were as 3 is to 4; find their present ages.

5 years ago the sum of the ages of *A* and *B* was 35 years; if 35 years be divided in the ratio of 3 to 4, the parts are 15 years and 20 years. ∴ The present age of *A* is (15+5) or 20 years, and that of *B* is (20+5) or 25 years.

Example 9. *A* is twice as old as *B*, and 4 years older than *C*; the sum of their ages is 71 years: find the age of each.

If *C* were as old as *A*, the sum of the ages of *A*, *B* and *C* would be 75 years; now, dividing 75 in the ratio of 2, 1 and 2, we find that the parts are 30, 15 and 30; ∴ *A*'s age is 30 years, *B*'s 15 years, and *C*'s (30−4) or 26 years.

Example 10. *A* and *B* begin business with equal capitals. At the end of the year *A* has gained ₨600, and *B* has lost $\frac{1}{10}$ of his capital; *A* has then twice as much as *B*. Find how much each had at first.

($\frac{9}{10}$ of *B*'s capital)×2=*A*'s capital+₨600,

∴ ($\frac{9}{10}$ of *A*'s capital)×2=.........................,

∴ $\frac{18}{10}$ or $1\frac{4}{5}$ of *A*'s capital=.........................,

i.e., *A*'s capital+$\frac{4}{5}$ of *A*'s capital=*A*'s capital+₨600,

∴ $\frac{4}{5}$ of *A*'s capital=₨600,

∴ *A*'s capital=₨600×$\frac{5}{4}$=₨750. *Ans.*

Example 11. Divide 250 into two parts such that, 3 times the first part and 5 times the second part may be together equal to 950.

3 times the 1st part + 5 times the 2nd part = 950; ...(i)

and the 1st part + the 2nd part = 250,

∴ 3 times the 1st part + 3 times the 2nd part = 750; ...(ii)

∴ 2 times the 2nd part = 200, [subtracting (ii) from (i)]

∴ the 2nd part = 100;

and ∴ the 1st part = 250 − 100 = 150.

Example 12. Mangoes are bought at R10 per 100; at what rate per 100 must they be sold that the gain on R100 may be equal to the selling price of 250 mangoes?

R100 is the cost price of 1000 mangoes; ∴ (1000 − 250) or 750 mangoes must be sold for R100; ∴ the selling price of 100 mangoes $= \text{R}100 \times \frac{100}{750} = \text{R}13\frac{1}{3}$.

Example 13. Two passengers going to the same place have 6 md. of luggage between them, and are charged for excess of luggage R4. 8*a.* and R3 respectively; had the luggage all belonged to one person he would have been charged R8. 4*a.* for excess. How much is allowed free?

R4 8*a.* + R3 is the charge on 6 md. *less* twice the free allowance, and R8. 4*a.* is the charge on 6 md. *less* the free allowance; ∴ the charge on free allowance = R8. 4*a.* − (R4. 8*a.* + R3) = 12*a.* ∴ (R8. 4*a.* + 12*a.*) or R9 = charge on 6 md.; ∴ 12*a.* = charge on $\frac{1}{2}$ md. Therefore $\frac{1}{2}$ md. is allowed free.

Example 14. Two guns are fired from the same place after an interval of 6 minutes, but a person approaching the place observes that 5 min. 51 sec. elapse between the reports; what was his rate of progress, sound travelling 1125 ft. per second?

In 5 min. 51 sec. or 351 sec. the man travels a distance which sound will travel in (6 min. − 5 min. 51 sec.) or 9 sec. But in 9 sec. sound travels 1125×9 ft.; ∴ in 351 sec. the man travels 1125×9ft.; ∴ in 1 hour the man travels $\frac{1125\times9\times60\times60}{351\times3\times1760}$ miles or $19\frac{191}{286}$ miles.

Example 15. R49 was divided amongst 150 children, each girl had 8*a.* and each boy 4*a.*; how many boys were there?

If 4*a.* be given to each child, R37. 8*a.* will be spent, and the boys will have got their shares. The remaining sum, R11. 8*a.*, must therefore be distributed amongst the girls only, giving 4*a.* to each. Hence the number of girls is the same as the *number of times* 4*a.* is contained in R11. 8*a.*; therefore the number of girls is 46, and therefore the number of boys is 104.

This example may also be solved by the method of Art. 225. Thus : When ₨49 is divided amongst 150 children, each gets $\frac{392}{75}a.$ on the average. Hence the question may be put thus—"Each boy is to have $4a.$ and each girl $8a.$; in what ratio should they be mixed that each may have $\frac{392}{75}a.$ on the average?" Therefore by the method of Art. 225 we find that the ratio of the number of boys to the number of girls must be $(8-\frac{392}{75}):(\frac{392}{75}-4)$ or $104:46$. But $104+46=150$; $\therefore$ the number of boys $=104$, and the number of girls $=46$.

Example 16. A free-hold estate is bought at 20 years' purchase; find the rate of interest obtained on the money invested.

["A free-hold estate is bought at 20 years' purchase" means that it is bought for 20 times the yearly rent derived from the estate.]

If the value of the estate is ₨20, the rent is ₨1 ; $\therefore$ if the value of the estate is ₨100, the rent is ₨5. Therefore the rate of interest obtained is 5 p. c.

Example 17. If 36 oxen in four weeks eat up the grass on a field of 12 acres and what grows upon it during the time ; and 21 oxen eat up the same in 9 weeks ; how many oxen will it maintain for 18 weeks, supposing the grass to grow uniformly during the time ?

Origl. growth + 4 wk.'s growth maintains 36 ox. for 4 wk.
$\therefore$ .... 1 ox for 144 wk. ;
also, origl. growth + 9 wk.'s growth.............. 21 ox for 9 wk.,
$\therefore$ .... 1 ox for 189 wk.

Hence, subtracting 2nd line from the 4th,

5 wk.'s growth maintains 1 ox for 45 wk.,
$\therefore$ 1 wk.'s growth .... 1 ox for 9 wk.,
$\therefore$ 16 wk.'s growth .... 1 ox for 144 wk.,
but origl. growth + 4 wk.'s growth .... 1 ox for 144 wk. ;
$\therefore$ origl. growth = 12 wk.'s.

Now, 1 wk.'s growth maintains for 9 wk. 1 ox,
$\therefore$ 1 wk.'s growth .... for 18 wk. $\frac{1}{2}$ ox,
$\therefore$ (12 + 18) or 30 wk.'s growth .... for 18 wk. 15 ox,
i.e., origl. growth + 18 wk.'s growth .... for 18 wk. 15 ox.

Answer. 15 oxen.

EXAMPLES. 173.

1. A person has a number of oranges to dispose of ; he sells half of what he has and one more to *A*, half of the remainder and

one more to *B*, half of the remainder and one more to *C*, and half of the remainder and one more to *D*: by which time he has disposed of all he had. How many had he at first?

2. A thief having stolen some money from the palace of Siraj Uddowlah was caught on his way back by the head *khoja* who let him off on getting half the money and ₹20 more; he was caught again by the sentry at the palace gate, who got a third of what he then possessed and ₹10 more; lastly he was let off by the *kotwal* in his rounds on getting $\frac{1}{4}$ of what he still had and ₹6 more. The thief came home robbed of all he stole. How much did he steal?

3. The expenses of a family, when rice is at 8 seers for a rupee, are ₹75 a month; when rice is at 10 seers for a rupee, the expenses are ₹72 a month (other expenses remaining unaltered): what will they be when rice is at 12 seers for a rupee?

4. A labourer was engaged for 15 days, on the agreement that for every day he worked he should have 6*a*., but that for every day he absented himself he would be fined 2*a*. He received ₹4. 2*a*. at the end of the time; how many days was he absent?

5. I have to be at a certain place in a certain time, and I find that if I walk 3 miles an hour I shall be 10 min. too late, and if I walk 4 miles an hour I shall be $7\frac{1}{2}$ min. too soon: what distance have I to go?

6. I have a certain sum of money to be distributed among a certain number of boys, and I find that if I give ₹2 to each I shall spend ₹4 too little, but if I give ₹3 to each I shall spend ₹3 too much. How much have I to spend?

7. I have a certain sum of money wherewith to buy a certain number of nuts, and I find that if I buy at the rate of 40 a penny I shall spend 5*d*. too much, if 50 a penny, 10*d*. too little. How much have I to spend?

8. A lb. of tea and 3 lb. of coffee cost 5*s*.; but, if coffee were to rise $33\frac{1}{3}$ p. c. and tea 50 p. c., they would cost 7*s*. Find the cost of tea and coffee per lb.

9. 3 lb. of tea and 4 lb. of sugar cost 8*s*.; but, if sugar were to rise 25 p. c. and tea were to fall 25 p. c., they would cost 7*s*. Find the cost of tea and sugar per lb.

10. Three tramps meet together for a meal: the first has 3 loaves, the second 4, and the third, who has his share of the bread, pays the other two 7 half-pence; how ought they to divide the money?

11. Two settlers in New Zealand own adjoining farms of 700 and 500 acres respectively. They unite their farms, taking at the same time a new partner who pays £1200 on the understanding that $\frac{1}{3}$ of the land will in future belong to each. How is the £1200 to be divided between the original owners?

12. The sum of the ages of A, B and C is now 90 years, and their ages 10 years ago were as $3 : 4 : 5$; find their present ages.

13. A is twice as old as B, and 5 years older than C; the sum of their ages is 45 years: find the age of each.

14. Divide ₨80 between A, B and C in such a manner that A may get 3 times as much as B, and B ₨10 more than C.

15. A and B begin business with equal capitals. At the end of the year A has gained ₨130, and B has lost $\frac{1}{15}$ of his capital; A has then twice as much as B. Find how much each had at first

16. A and B begin business with equal capitals. At the end of a certain time A has gained $\frac{1}{4}$ of his capital, and B has lost ₨200; B has now $\frac{1}{3}$ of what A has. How much had each at first?

17. Divide 155 into two parts such that, twice the first part and 3 times the second part may be together equal to 370.

18. Divide 100 into two parts such that, $\frac{1}{2}$ of one part and $\frac{1}{3}$ of the other part may be together equal to 40.

19. Divide 350 into two parts such that, 3 times the first part and $\frac{1}{3}$ of the second part may be together equal to 250.

20. Mangoes are bought at ₨5 per 100; at what rate per 100 must they be sold that the gain on ₨100 may be equal to the selling price of 400 mangoes?

21. Sugar is bought at 4*a.* per seer; at what rate per seer must it be sold that the gain on ₨10 may be equal to the selling price of 8 seers?

22. Two passengers going to the same place had 8 md. of luggage between them, and were charged for excess of luggage ₨8 and ₨4 respectively; had the luggage all belonged to one person he would have been charged ₨14 for excess. Find how much is allowed free, and how much luggage each had.

23. Two guns are fired from the same place after an interval of 10 minutes, but a person approaching the place observes that 9 min. 30 sec. elapse between the reports; what was his rate of progress, sound travelling 1121 ft. per second?

24. Two guns are fired from the same place at an interval of 15 minutes, but a person going away from the place hears the reports at an interval of 15 min. 30 sec.; if sound travels 1125 ft. per second, find his rate of travelling per hour.

25. Two guns are fired from a place at an interval of 28 minutes, but a person approaching the place, at the rate of $13\frac{10}{11}$ miles an hour, hears the reports at an interval of 27 min. 30 sec. Find the velocity of sound per second.

26. Cannons are fired at regular intervals in a town, and a person riding towards it at the rate of 9 miles an hour hears the

reports at intervals of 15 minutes; at what intervals must the cannons have been fired, sound travelling 1120 ft. per second?

27. Cannons are fired at intervals of 10 minutes in a town towards which a passenger train is approaching at the rate of 30 miles an hour; if sound travels 1136 ft. per second, find at what intervals the reports will be heard by the passengers.

28. Rs.60 was distributed among 50 children, each girl had Rs.2 and each boy Rs.1; how many boys were there?

29. 35 fruits, consisting of mangoes and oranges, were bought for Rs.2. 8*a.*; if the mangoes cost 2*a.* each and the oranges 6*p.* each, find the number of oranges bought.

30. A lump composed of gold and silver measures 6 cu. inches and weighs 100 oz.; if a cu. inch of gold weighs 20 oz. and an equal bulk of silver 12 oz., find the weight of gold in the mixture.

31. 19 grains of gold or 12 grains of silver displace one grain of water. If a ring, composed of gold and silver, weighs 88 grains and displaces 5 grains of water, how many grains of silver does it contain?

32. A farmer has oxen worth £12. 10*s.* each, and sheep worth £2. 5*s.* each; the number of oxen and sheep being 35, and their value £191. 10*s.* Find the number he had of each.

33. If an income-tax of 7*d.* in the £ on all incomes below £100 a year, and of 1*s.* in the £ on all incomes above £100 a year realises £18750 on £500000, how much is raised on incomes below £100 a year?

34. How many years' purchase should be given for a free-hold estate so as to get 5 per cent. for the money?

35. An estate is bought at 25 years' purchase for Rs.40,000, one-fourth of the purchase-money remaining at mortgage at 6 per cent. The cost of collecting rents is Rs.100 per annum. What interest does the purchaser make on his investment?

36. If 10 oxen in 5 weeks eat up the grass on a field of 7 acres and what grows upon it during the time, and 11 oxen eat up the same in 4 weeks, how many weeks' growth is on the field?

37. If 20 oxen in 4 weeks eat up the grass on a field of 4 acres and what grows upon it during the time; and 17 oxen eat up the same in 10 weeks; how many oxen will it maintain for 5 weeks, supposing the grass to grow uniformly during the time?

38. In a certain meadow there is a crop of 525 stones of grass, which grows uniformly. If 11 oxen turned in would consume all the grass in 48 days, but 6 oxen would require 98 days, what weight of grass would each ox eat in a day?

39. If 25 horses eat the grass of 35 acres of one field in 11

days, in what time would 20 horses eat the grass of another field of 56 acres, where there is at first twice as much grass per acre as in the former field, the growth of the grass being neglected. What must be the ratio of the rates of the growth of the grass in the two fields so that your result may be accurately true?

40. A well is fed by a spring which flows continuously and uniformly into it. When there are 10,000 cu. ft. of water in the well, 7 men can empty it in 20 days; and when there are 15,000 cu. ft. of water in the well, 5 men can empty it in 50 days. How many cu. ft. of water flow into the well in one day?

41. A cistern has one supply-pipe (*A*) and 2 equal waste-pipes (*B*, *C*) attached to it. *A* is opened, and when the cistern is partially filled *B* is also opened, and the cistern is emptied in 3 hours. Had *C* been opened along with *B* the cistern would have been emptied in 1 hour. How long after *A* was *B* opened?

42. A cistern has two pipes attached to it, one to supply and one to draw off. If both the pipes are opened together, the cistern is filled in 9 hours; but if the waste-pipe is opened one hour after the supply-pipe, the cistern is filled in 7 hours. In what time can the supply-pipe fill the empty cistern?

43. A leaky cistern is filled in 5 hours with 30 pails of 3 gallons each, but in 3 hours with 20 pails of 4 gallons each, the pails being poured in at intervals. Find how much the cistern holds, and in what time the water would waste away.

EXAMPLES FOR EXERCISE. 174a.

(First Series.)

1. State in words 10030200720021.

2. Find the value of $66674-9645-201+843-8761$.

3. Reduce £49. 6*s*. $2\frac{1}{4}$*d*. to farthings.

4. Find the prime factors of 51425.

5. Reduce $\frac{18577}{20006}$ to its lowest terms.

6. Find the sum and difference of 23·001 and ·0414.

7. Find the value of $\frac{3}{8}$ of ₹7. 7*a*. 7*p*.

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8. Write in words 3200103102 according to the Indian numeration.

9. The greatest prime number known is expressed by 1251^2+2920^2; find this number.

10. What sum will remain when four bills, amounting to ₹5 . 7 . 6, ₹3. 4 . 9, ₹2 . 15 . 3, and ₹10 . 13 . 3 respectively, have been paid out of ₹25?

11. Find the G. C. M of 23791 and 8029.

12. Subtract $14\frac{51}{140}$ from $16\frac{4}{35}$.

13. Multiply ·038 by ·0042, and divide ·03217 by 6·25.

14. Find the value of ·00625 of £1.

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15. Subtract one crore five lacs three thousand and twenty from twenty-nine million twelve thousand and four.

16. Multiply 765389 by 64164 in 3 lines.

17. I go to town with £9. 1*s*. 3*d*. What have I left after buying a dozen chairs at 13*s*. $7\frac{1}{2}$*d*. each ?

18. Find the L. C. M. of 9569 and 16115.

19. Add together $\frac{11}{17}$, $3\frac{1}{9}$, $1\frac{7}{81}$ and $\frac{7}{27}$.

20. Express as a decimal ·0003 + [illegible] − ·00849 + [illegible].

21. Reduce $\frac{2}{9}$ of $\frac{5}{13}$ of 19*s*. 6*d* to the fraction of [illegible] of $\frac{4}{17}$ of £1. 8*s*. 4*d*.

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22. Express 944 in Roman notation, and CDXCIX in Arabic notation

23. Multiply 387659 by 85672 in 3 lines.

24. How many cows at ₨10. 14*a*. each can I buy with the proceeds of selling 87 horses at ₨115. 2*a*. each ?

25. Simplify $\dfrac{6\frac{3}{4} - 1\frac{5}{7}}{2\frac{1}{6} + 1\frac{3}{7}}$.

26. Multiply ·000134 by 80·032, and divide the result by ·0032.

27. Reduce $(8 \div 1\frac{1}{4})$ of 1*p*. to the decimal of ₨1. 4*a*.

28. If a rupee is worth 2*s*. $0\frac{1}{2}$*d*., and a dollar 4*s*. $4\frac{1}{2}$*d*., find the least number of rupees which makes an exact number of dollars.

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29. What number multiplied by 76 will give the same product as 153 multiplied by 380 ?

30. Find the greatest number which will divide each of 3456, 26244 and 99225 without remainder.

31. Reduce 57 tons 9 cwt. 1 qr. 10 lb. to drams.

32. Simplify $\frac{3}{4} \times \frac{5}{8} \div 1\frac{1}{3}$ of $1\frac{1}{2}$.

33. Find the least fraction which being added to $\frac{1}{3} - \frac{1}{4}$ of $\frac{1}{5} - \frac{1}{6}$ will make the sum an integer.

34. *A* did ·0025 of a piece of work, and *B* ·7855. How much was left undone ?

35. Find the cost of 3·125 yards at £·375 a yard.

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36. What number is the same multiple of 35 that 3456 is of 9?

37. If my income is R3500 and I save R507 a year, what is my average daily expenditure?

38. Simplify $\dfrac{(\frac{1}{2}-\frac{1}{3}) \text{ of } (\frac{1}{8}-\frac{1}{9})}{\frac{1}{2}-\frac{1}{3} \text{ of } \frac{1}{8}-\frac{1}{9}}$.

39. If the sum of $21\frac{5}{7}$ and $3\frac{6}{11}$ be added to the product of $2\frac{3}{4}$ and $\frac{7}{8}$, by how much will the result differ from 28?

40. Reduce $3\frac{17}{247}$ to a decimal.

41. Find the vulgar fraction equal to $\cdot 27\dot{8}9\dot{9}$.

42. Find the value of $\frac{3}{8}$ of R3 . 7 . 6 + ·375 of R6 . 8 . 6.

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43. Find the least number which being subtracted from 97856 will make the result divisible by 141.

44. Reduce 3 acres 1 rood 2 perches to square feet.

45. Arrange $\frac{2}{5}, \frac{3}{7}, \frac{27}{70}$ in order of magnitude.

46. Divide $\frac{2}{3} \div \frac{3}{4}$ of 12 by $\frac{2}{3}$ of $\frac{3}{4} \div 12$.

47. Add $3\cdot 7\dot{2}\dot{5} + \cdot 00\dot{2} + \cdot 27\dot{2}\dot{5}$.

48. Reduce $\cdot 0\dot{3}$ of R3 to the decimal of $\frac{3}{4}$ of R1·5.

49. Find the least number of weeks in which an exact number of half-guineas can be earned, the wages per week being 7·5 shillings.

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50. What is the least number which being added to 30321 will make the sum divisible by 681?

51. A bill of £6 1*s.* 11*d.* has to be paid by several persons in equal shares; if three of them together pay £1. 13*s.* 3*d.*, how many are there to share the cost?

52. Simplify $2\frac{10}{17} \times 1\frac{33}{35} \div \frac{87}{28} \times 2\frac{13}{66}$.

53. Divide 352·95624 by ·000504.

54. Express $1\cdot 4 \div 1\cdot \dot{1}\dot{3}$ as a decimal.

55. Reduce ·543 of 19*s.* $3\frac{1}{4}$*d.* to pence.

56. Find the greatest unit of time by means of which 2 hr. 3 min. and 1 hr. 4 min. 30 sec. can both be expressed as integers.

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57. I multiply a number by 36 and divide the result by 12 and obtain 374181 as quotient. What was the number?

58. *A* and *B* together have R36. 13*a.* 9*p.*, and *A* has R3. 3*a.* 3*p.* more than *B*; find how much *B* has.

59. Reduce $\frac{5184}{19008}$ to its lowest terms.

60. Express $3\frac{19}{22}$ poles in poles, yards, etc.

61. What are the nearest integers to $8\frac{9}{16}$ and $7\frac{6}{11}$?

62. Find the difference between the product and quotient of $5\cdot312$ by $\cdot0125$.

63. Simplify $(2\cdot3\dot{6}\dot{4}-1\cdot6\dot{9}\dot{7})+1\cdot\dot{3}\times(2\cdot\dot{4}+7\cdot\dot{5})$.

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64. If in a division sum the divisor be 7 times and the quotient 5 times the remainder, what is the dividend when the remainder is 360?

65. Reduce 300,003,840 grains to pounds Troy.

66. Find the cost of 13724 articles at R8. 0*a*. $7\frac{1}{2}$*p*. each.

67. Multiply $7\frac{1}{6}+6\frac{2}{3}$ by $2\frac{1}{13}-\frac{23}{52}$.

68. What fraction of a journey of 15 miles have I gone on reaching a place $6\frac{2}{3}$ miles distant?

69. By what must $1550\frac{13}{16}$ be divided that the quotient may be $459\frac{1}{2}$?

70. If a metre be $39\cdot37$ inches, how many metres make 3 miles?

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71. When 2080400 is divided by a certain number, the quotient is 381 and the remainder 1664. What is the number?

72. Reduce 67501 inches to poles, etc.

73. If $2\frac{1}{16}$ tons cost R954. 3*a*. 8*p*., what is the cost of 1 ton?

74. Simplify $\frac{3-4\frac{1}{7}+2\frac{1}{3}}{3\times2\frac{1}{3}-4\frac{1}{7}}\div\frac{6\frac{1}{7}\text{ of }4\frac{4}{5}}{11\frac{1}{6}-6\frac{7}{10}}$.

75. Divide equally amongst 5 boys $\frac{4}{7}$ of £4. 2*s*. $1\frac{3}{4}$*d*.

76. Divide $\cdot7029$ by $\cdot0165$.

77. What decimal of R3. 7*a*. must be taken from R4. 15*a*. to leave R$2\cdot5$?

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78. If when a number is divided continuously by 5, 6 and 7, the remainders are 2, 3 and 4 respectively, what would be the remainder if the number were divided by 210?

79. If 1 md. cost R11. 1*a*., find the cost of $\frac{70}{531}$ of a md.

80. The 1st of January 1893 was on a Sunday; on what day of the week will 10th February fall in the year 1894?

81. Find the value of $\frac{7\frac{6}{11}}{8\frac{7}{22}}\div\frac{2\frac{5}{8}}{7\frac{1}{2}}\text{ of }\frac{81\frac{4}{11}}{5\frac{1}{12}}$.

82. If from a rope 7 ft. long as many pieces as possible are cut off, each $1\frac{1}{8}$ ft. long, what fraction of the whole will be left?

83. Reduce $\cdot\dot{1}4285\dot{7}+\cdot\dot{8}5714\dot{2}-\cdot\dot{2}8571\dot{4}$ to a vulgar fraction.

84. Simplify $\frac{1\cdot5}{\cdot075}\times\frac{3\cdot25}{1\frac{1}{2}}$.

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85. Find a number such that if it be added 35 times to 25 the sum will be 25540.

86. If a person spends in 4 months as much as he earns in 3, how much can he lay by annually, supposing that he earns £250. 10*s*. every 6 months?

87. Simplify $\frac{(3\frac{1}{3}-2\frac{1}{2})\div\frac{5}{6}\text{ of }\frac{3}{8}}{2\frac{2}{3}\div(\frac{1}{2}+\frac{1}{4})}$.

88. How many steps does a man whose length of pace is 32 inches take in $4\frac{3}{4}$ miles?

89. Divide ·75445 by ·00625.

90. How many inches are there in ·1215625 of a mile?

91. Subtract ·432 of an acre from $2\frac{1}{2}$ roods, expressing the result in sq. yards and the decimal of a sq. yard.

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92. A man buys 100 md. of rice; he loses as much by selling 60 md. at ₨3 a md. as he gains by selling the rest at ₨4. 4*a*. a md. Find the cost price of a md.

93. By what prime numbers may 109 be divided so that the remainder may be 4?

94. Add $\frac{6479}{6510}+\frac{4347}{4410}+\frac{6831}{6930}$.

95. How many times can ·053 be subtracted from 14·578, and what will be the magnitude of the remainder?

96. Express $\cdot2\dot{3}\dot{6}$ of 4*a*. 7*p*. + $\cdot51\dot{6}$ of 10*a*. as the decimal of ₨1. 4*a*.

97. Simplify $\frac{(3\cdot2-2\cdot9)\times147}{\cdot003\times\cdot0005}$.

98. Three bells toll at intervals of 1·2, 1·8 and 2·7 seconds respectively, beginning together; how often will each toll before their tolling together again?

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99. The remainder after a division is 97, the quotient is 521, and the divisor is 9 more than the sum of both; what is the dividend?

100. Two pieces of cloth of the same length cost £5. 11*s*. 9*d*. and £7. 4*s*. respectively; the price of the first was 3*s*. $1\frac{1}{2}$*d*. per yard: what was the price of the second per yard?

101. Divide $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{5}{7}$ of 42 by the sum of $2\frac{1}{3}$ and $4\frac{5}{6}$.

102. Simplify $\frac{1}{2}[2-\frac{1}{2}\{2-\frac{1}{2}(2-\frac{1}{2})\}]$.

103. Reduce $\frac{29}{66}$ to a decimal.

104. Multiply 28·8 by 25·3 and divide the product by 6·48.

105. The distance between two wickets was marked out for 22 yd., but the yard measure was $\frac{6}{12}$ of an inch too short : what was the actual distance ?

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106. If a number of articles at R4. 0*a*. $5\frac{1}{4}$*p*. each cost R7059. 14*a*. $11\frac{1}{4}$*p*., how many are there ?

107. Simplify $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{2}+\frac{1}{3}}$ of $\dfrac{\frac{1}{4}-\frac{1}{6}}{\frac{1}{4}+\frac{1}{6}}$ of $\dfrac{\frac{1}{6}-\frac{1}{7}}{\frac{1}{6}+\frac{1}{7}}$ of 117.

108. Find the value of $\dfrac{\cdot 426\times\cdot 426-\cdot 174\times\cdot 174}{\cdot 426-\cdot 174}$ of R1. 4*a*.

109. Subtract 5·142857 from $5\cdot\dot{1}4285\dot{7}$.

110. Divide 1·00625 by 132·5 to five places of decimals.

111. Reduce 4 hr. 48 min. to the decimal of 6 hr.

112. A man owns $\frac{3}{16}$ of a house, and sells $\cdot 1\dot{3}5\dot{1}$ of his share ; what fraction of the house does he still own ?

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113. How many revolutions will be made by a wheel which revolves at the rate of 243 revolutions in 3 min., while another wheel revolving 374 times in 11 min. makes 544 revolutions ?

114. Multiply 10 sq. yd. 4 ft. 76 in. by 132.

115. Reduce to its lowest terms [illegible].

116. Find the least number which, when divided by each of $\frac{7}{12}$, ·25, and $\cdot 0\dot{3}$, gives a whole number as quotient in each case.

117. Simplify $\dfrac{5\cdot 3\dot{4}\times 5\cdot 3\dot{4}-2\cdot 6\dot{5}\times 2\cdot 6\dot{5}}{5\cdot 3\dot{4}-2\cdot 6\dot{5}}$.

118. Find, to the nearest pie, the value of ·1234 of R12·5.

119. A kilolitre contains 35·32 cubic feet, and a gallon contains 277·274 cubic inches ; find to the nearest integer the number of gallons in a kilolitre.

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120. A farmer has 899 sheep and 493 lambs. He forms them into flocks, keeping sheep and lambs separate, and having the same number of animals in each flock. If these flocks are as large as possible, how many flocks will there be altogether ?

121. If 257 pounds of tea cost £34. 16*s*. $7\frac{1}{2}$*d*., find the price of a pound to the nearest farthing.

122. Simplify $\dfrac{\frac{1}{2}\div\frac{1}{3}}{\frac{1}{4}\div\frac{1}{5}}\div\dfrac{\frac{1}{6}\div\frac{1}{7}}{\frac{1}{8}\div\frac{1}{9}}$.

123. How many whole cakes will be required for 50 children if each is to have $2\frac{2}{7}$ of $1\frac{4}{11}$ of $2\frac{5}{8}$ of $\frac{4}{9}$ of $\frac{11}{16}$ of $\frac{1}{10}$ of a cake ?

124. Find the value of $\dfrac{\frac{3}{8} \text{ of } \cdot375-\frac{2}{45} \text{ of } \cdot0\dot{4}}{\cdot375+\cdot0\dot{4}}$.

125. Find the circulating decimal which will become 2 when multiplied by $2\frac{3}{8}\div4\cdot5$.

126. A German mark is worth £·04895 ; find to the nearest farthing the value of 3725·39 marks.

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127. To a certain number I add 2, I multiply the sum by 4, I divide the product by 3, and I take 3 from the quotient ; the remainder is 17. What is the number ?

128. On what day of the week will Feb. 10 fall in the year 1960 ?

129. Find the greatest prime number which used as divisor of 12260 will leave remainder 17.

130. Find the value of $\dfrac{2\cdot8}{\cdot21}$ of $\dfrac{\text{R}1.\ 5a.\ 4p.}{\text{R}4.\ 2a.\ 8p.}$.

131. What is the number whose half exceeds its fifth part by 6 ?

132. Simplify $\cdot\dot{4}2857\dot{1}\times\cdot49\times\cdot20\dot{5}71428\dot{8}$.

133. How many times does a carriage wheel, whose circumference is 17·125 feet, turn round in a distance of $12\cdot\dot{4}\dot{5}$ miles ?

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134. Determine the prime factors of 282660 and 40299. Hence deduce the G. C. M. and L. C. M. of these numbers.

135. Find the least integer which, when divided by $1\frac{5}{21}$ and $1\frac{6}{7}$, will give a whole number as quotient in each case.

136. Simplify $\frac{3}{7}$ of $\dfrac{6\frac{5}{12}}{3\frac{2}{3}}-\frac{19}{20}$ of $\dfrac{1\frac{2}{3}}{6\frac{2}{3}}+\frac{3}{5}(\frac{3}{2}\times\frac{1}{6}+\frac{5}{8}\div\frac{1}{3})$.

137. Reduce $\frac{5}{99}+\frac{7}{999}+\frac{16}{99999}$ to a decimal.

138. If a cu. yd. of clay make 460 bricks, each $101\frac{1}{4}$ cu. in., how much does clay contract in baking ?

139. Multiply 324·567 by 13·212 in 2 lines.

140. One pendulum oscillates 6 times in 3·2 seconds, and another pendulum 8 times in 3·6 seconds ; if started simultaneously, how often will they tick together in an hour ?

EXAMPLES FOR EXERCISE. 174. b

(*Second Series.*)

1. Write down the greatest and least numbers of four digits that you can form with the figures, 3, 0, 2, 1.

2. Simplify $\frac{1}{3}[3+\frac{1}{3}\{3+\frac{1}{3}(3+1\frac{1}{2})\}]\div\frac{1}{8}$.

3. The telegraph posts on a railway line are 66 yards apart; find the smallest number of miles that corresponds to an exact number of posts.

4. A bath is supplied with water from two pipes, one of which can fill it in $12\frac{1}{2}$ min., the other in 15 min.; there is also a discharging pipe which would empty it, when filled, in 10 min. The first pipe is open alone for 4 min., and then the first and second open together for 1 min.; if now the third pipe is opened as well, how long will it take to fill the bath?

5. The wages of A and B together for 20 days amount to the same sum as the wages of A alone for 35 days. For how many days will this sum pay the wages of B alone?

6. A cask contains 5 parts wine and 3 parts water; how much of the mixture must be drawn off and water substituted in order that the resulting mixture may be half and half?

7. A person borrows £130 on the 5th of March, and pays back £133. 18*s.* on the 10th October; find the rate of interest charged.

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8. The digits in the units' and lacs' places of a number are 3 and 8 respectively; what will be the digits in the same places in the remainder when 99999 is subtracted from the number?

9. A whole number diminished by $\frac{1}{3}$ of itself, when divided by 307 gives a quotient 12 and a remainder 95; what is the number?

10. The length of a rectangular tennis-court is 5 yards longer than its breadth, and its perimeter is 130 yards; find its area.

11. The train which leaves Calcutta at 4-30 P. M. arrives at Burdwan at 8 P. M.; and the train which leaves Burdwan at 4-50 P. M. arrives in Calcutta at 8-30 P. M.: when do they pass each other?

12. The rent of a farm consists of a fixed sum of money together with the value of a certain number of maunds of wheat; when wheat is ₨2. a md. the rent is ₨40; when wheat is ₨2. 4*a.* a md. the rent is ₨42. 8*a.* What will be the rent when wheat is ₨2. 10*a.* a md.?

13. Assuming that the circumference of a circle is to its diameter as 22 is to 7, and that the circumference of the earth is to its diameter as 160 metres to 167 feet, determine to 4 places of decimals the ratio of a metre to a foot.

14. The interest on a given sum of money for one year is £5. 8s. 4d., the compound interest for two years is £11. 1s. Find the rate per cent.

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15. If when a number is divided continually by 5, 6 and 8 the remainders are 2, 3 and 4 respectively, what would be the remainder if the same number were divided by 240?

16. Divide 1255 by 1·004, and hence deduce the quotient of 12·55 by 1004 and ·01255 by 1004000.

17. I bought a certain number of chairs for ₨45; also a certain number for ₨28. 2a. at the same rate: find the greatest possible price of each chair.

18. A clock which gains $2\frac{1}{4}$ min. in a day, is 3 min. slow at noon on Sunday; when will it show correct time, and what time will it indicate at 6 on Monday evening?

19. A person bought 4 railway tickets to go 60 miles. Two were for the 1st class, one for the 2nd, and the fourth, a half first class ticket, for a child. The cost of a 2nd class ticket was $\frac{2}{3}$ of that of a first class, and the whole sum paid was £1 . 11 . 8. Find the price of each ticket and the rate per mile for the first class.

20. There are two mixtures of wine and water, in the ratios of 3 : 2 and 4 : 5 respectively; if one gallon of the first be mixed with 2 gallons of the second, what fraction of the resulting mixture will be wine?

21. A book sent from England costs me (including 1s. 6d. postage) 16s. 1d., my book-seller allowing me two pence in the shilling discount on the published price. What is the published price?

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22. What number is the same multiple of 7 that 3975 is of 15?

23. Simplify $\frac{1}{7\frac{1}{4}+6\frac{6}{11}} \div \left(\frac{3}{13}-\frac{2}{9}\right) - \left(\frac{13}{3}+\frac{1}{6}\right) \div \frac{2}{3}$ of $\frac{3}{8}$ of 63.

24. On laying down a bowling-green with sods 2 ft. by 9 in., it is found that it requires 120 sods to form one strip extending the whole length of the green, and that a man can lay down one strip and a half each day; find the space laid down by 5 men in 2 days.

25. *A* can do a piece of work in 3 days, *B* can do 3 times as much in 8 days, and *C* 5 times as much in 12 days. In what time will they do it together, supposing them to work at the rate of 9 hours a day?

26. A farmer pays a corn-rent of 5 quarters of wheat and 3 quarters of barley, Winchester measure; what is the money value of his rent, when wheat is at 60s., and barley at 54s. per quarter, Imperial measure; 32 Imperial gallons being equal to 33 Winchester gallons?

27. Six coins of equal weight, made of gold and silver mixed, were melted together and re-cast. In one the gold and silver were in the ratio of 2 : 3; in two others, of 3 : 5; and in the rest, of 5 : 4. In what ratio will the gold and silver be mixed in the new coins?

28. A tradesman, selling goods for a certain price to be paid six months hence, offers to give one-tenth more of the same goods for the same price in ready money. What is the rate of discount?

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29. Find the greatest and least numbers of 6 digits which are exactly divisible by 239.

30. There is a number, to which 3 is added and $\frac{1}{10}$ of the result taken; to this 5 is added and $\frac{1}{18}$ of the result taken, giving $1\frac{1}{2}$: what is the number?

31. Find all the numbers of 5 digits divisible by 9, which have unity for their first and last digits and 2 for their middle digit. State the principle upon which you proceed.

32. On a stream, B is intermediate to and equidistant from A and C; a boat can go from A to B and back again in 5 hr. 15 min., and from A to C in 7 hr. How long would it take to go from C to A?

33. If the price of bricks depends upon their magnitude, and if 100 bricks, of which the length, breadth and thickness are 16, 10 and 8 inches respectively, cost R2. 9a., what will be the price of 921600 bricks which are one-fourth less in every dimension?

34. There are two mixtures of wine and water, the quantities of wine in them being respectively ·25 and ·75 of the mixtures. If 2 gallons of the first be mixed with 3 gallons of the second, what will be the ratio of wine to water in the compound?

35. How much per cent. must be added to the cost price of goods that a profit of 20 per cent. may be made after throwing off a discount of 10 per cent. from the labelled price?

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36. Determine the least number, by which 616 must be multiplied so as to produce a number exactly divisible by 770.

37. Multiply the sum of $2\cdot\dot{4}$ and $7\cdot\dot{5}$ by $1\cdot\dot{3}$, and add the result to the difference of $2\cdot36\dot{4}$ and $1\cdot\dot{6}9\dot{7}$.

38. The floor of a room is 50 ft. long and 40 ft. wide. Find the cost of supplying it with carpet, 2 ft. wide, at R3 per yard, and oil-cloth, 2 yards wide, at R1 per yard; the oil-cloth to be laid along the sides and ends a yard and a half wide, and the carpet to extend one foot over the oil-cloth everywhere.

39. On a certain evening half an hour after sunset a watch was set at 12 o'clock. The morning following it was 8 minutes

past 4 by a common clock when it was 4 minutes past 8 by this watch. Find the time of sunset the previous evening.

40. *A* has shares in an estate to the amount of $(\cdot 15 \div \cdot 3\dot{6})$ of it. *B* has shares in the same estate to the amount of $\cdot 4\dot{7}\dot{2}$ of it. Find the difference in value between the properties of *A* and *B*, when $\cdot 0\dot{5}\dot{6}$ of the estate is worth £373·3.

41. Three equal glasses are filled with mixtures of spirit and water : the proportion of spirit to water in each glass is as follows : in the first glass as 2 : 3, in the second glass as 3 : 4, and in the third as 4 : 5. The contents of the three glasses are emptied into a single vessel ; what is the proportion of spirit and water in it ?

42. If the true discount on a bill of £14641 be £4641 at 10 per cent. compound interest, how many years has the bill to run ?

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43. Twenty-fifth part of a certain number is equal to the seventh part of 42 ; what is the number ?

44. Simplify $\frac{91}{122}(4\frac{1}{3} \text{ of } 6\frac{2}{7}+\frac{3}{14}) \div 4\frac{1}{3} \text{ of } (6\frac{2}{7}+\frac{3}{14})$.

45. A company of Sepoys proceed in 5 equal rows, and after sometime arrange themselves into 7 equal rows. Find the least number above 1000, which the company may contain.

46. *A* is twice and *B* is just as good a workman as *C*. The three work together for two days, and then *A* works alone for half a day, and *B* for a day. How long would it have taken *A* and *C* together to complete as much as the three will have thus performed ?

47. A steam-ship whose speed averages 14 miles an hour, reaches a certain port in 12 days ; how many days afterwards will a sailing vessel arrive, which started at the same time and sailed on an average 8 miles an hour ?

48. From a cask of wine $\frac{1}{5}$ is drawn off and the cask is filled up with water ; $\frac{1}{5}$ of the mixture is then drawn off and the cask is again filled up with water ; after this process has been repeated 4 times, what will be the ratio of wine to water in the resulting mixture ?

49. The sum of £2100 is due in 4 years, but it is paid by instalments as follows :—£275 at the end of 2 years, £460 at the end of the 3rd year, £500 at the end of the 4th year, and £600 at the end of the 5th year. What amount should be paid at the end of the 6th year, in order to clear off the balance, simple interest being reckoned at the rate of 5 per cent. per annum ?

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5 . Twenty times a certain number is equal to 7 times 40 ; what is the number ?

51. What is the least number of shot, each $1\frac{2}{3}$ oz., that will weigh an integral number of pounds ?

✓ 52. A *rod of brick work* contains 306 cu. ft.; find the cost of building a brick wall, 68 yd. by 6 ft. by 2 ft. 2 in., at R18 per rod.

✓ 53. How long would a column of men, extending 3420 feet in length, take to march through a street, a mile long, at the rate of 58 paces in a minute, each pace being $2\frac{1}{2}$ feet?

✓ 54. 195 men are employed to work on a railway embankment, $1\frac{1}{2}$ miles long, which they are expected to finish in 4 weeks. But at the end of 1 week it is found that they have finished only 520 yards. How many more men must be engaged to finish it in the required time?

55. *A* is a cask containing 125 gallons of wine; *B* is another cask containing 175 gallons of water. 100 gallons are drawn from each, mixed together, and the casks are refilled with the mixture. This operation is once more repeated. Find the ratio of wine to water in each cask now.

✓ 56. A person who pays 5*d*. in the £ income-tax finds that a rise of interest from 6 to $6\frac{1}{2}$ per cent. increases his income by £23. 10*s*. What is his capital?

✓ 57. From a certain number I take 320; to the remainder I add 24; I multiply the sum by 8, and find that the product is equal to the sum of 304 and 760: what is the number?

✓ 58. What decimal of 2·25 units is ·05 of a unit?

✓ 59. A jar can be exactly filled by glasses holding 3 pints each; it can be exactly emptied again by glasses holding 5 pints each; given that the capacity of the vessel is between 11 and 12 gallons, find the exact capacity.

60. Two clocks are set right at noon on Monday. One loses and the other gains 1 min. a day. What time will be indicated by the latter, when the former points 10 h. $49\frac{1}{11}$ m. P. M. on the following Saturday?

✓ 61. Three gardeners working all day can plant a field in 10 days, but one of them having other employment can work only half time. How long will it take them to complete the work?

62. One vessel contains 20 gallons of wine; another contains 20 gallons of water. One gallon is taken from each, and poured into the other. This is done 3 times. Find the strength of the two mixtures.

63. A gentleman bequeaths his property to his children to be so divided that their shares shall be equal on their coming to age at 21, counting interest and discount at 5 per cent. He dies worth £13240, leaving three children aged 23, 21 and 19 respectively. How much should each receive?

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64. To a certain number I add 7, I multiply the sum by 5, I divide the product by 9, and take 3 from the quotient; the remainder is 12: what is the number?

65. Simplify $$(\cdot 5+\cdot 75)(2\cdot 5-\cdot 4)\div\left(\cdot 125+\frac{1}{4\cdot 8}\right).$$

66. Find the weight in tons per sq. mile of a rain-fall of 7 inches, having given that a cu. ft. of water weighs 1000 oz.

67. *A*, *B* and *C* are employed on a piece of work. After 15 days *A* is discharged, $\frac{1}{3}$ of the work being done. *B* and *C* continue at the work, and after 20 days more *B* is discharged, $\frac{1}{3}$ more of the work being done. *C* finishes the work in 30 days. In what time would the work have been done, if *A* and *B* had continued to work?

68. If one man walks 165 miles in 6 days, how far will another man walk in 15 days, if the first man walks $3\frac{3}{4}$ miles in the same time that the other man takes to walk 4 miles?

69. If 3 cubic inches of iron and 2 cubic inches of water weigh as much as 2 cubic inches of iron and 9 cubic inches of water; find the ratio of the weight of a cubic inch of iron to that of a cubic inch of water.

70. I buy goods for R600, and sell them directly for R680, giving three months' credit; what is gained per cent. per annum?

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71. From the tenth part of a certain number I subtract 10, and find that the remainder is 10; what is the number?

72. $\frac{4}{5}$ of a number exceed the sum of its third and fourth parts by 26; what is the number?

73. Two cog-wheels, having 75 and 130 teeth respectively, are working together; after how many revolutions of the smaller wheel will the teeth which once touch, touch again?

74. A train leaves *P* for *Q*, at the same time that a train leaves *Q* for *P*; the trains meet at the end of 6 hours, the train from *P* to *Q* having travelled 8 miles an hour more than the other. Find the rates of the trains, the distance from *P* to *Q* being 162 miles.

75. If 1000 rupees a month be equivalent to £1112. 10*s*. a year, what is the value of a rupee in English money?

76. Divide £20 among 2 men, 3 women and 4 children, so that each woman gets twice as much as a child, and each man as much as a woman and a child together.

77. If the interest of £253. 2*s*. 6*d*. at 5 p. c. be equal to the discount on £257. 6*s*. 10½*d*. for the same time and at the same rate, when is the latter sum due?

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78. Find a number such that if it be subtracted 25 times from 7201 the remainder will be 951.

79. How many parcels of gold dust, each weighing 17·35 grains, can be made up out of 1 lb. 2 oz. 1 dwt. 3 gr.; and how much will remain over?

80. A room is 20 ft. long, 15 ft. wide and 10 ft. high. There are in it 4 doors, each 7 ft. by 4 ft.; the fireplace is 6 ft. wide and 4 ft. high; a skirting 2 ft. deep runs round the walls. Find the expense of papering the room at 6 annas a sq. yd.

81. If the hands of a clock coincide every $65\frac{1}{2}$ min. (true time), how much does the clock gain or lose in a day?

82. *A* can copy a certain manuscript in 17 hours by writing at the rate of 3 lines per minute; *B* can copy the same in 24 hours. After 476 lines have been copied by *A*, in what time can *B* finish it?

83. A town contains 12 Hindus to every 3 Mahomedans and to every 3 Christians; if there are 4800 Hindus, find the number of Christians.

84. Two sums, each of £138. 2*s*. 6*d*., being due, one at the present time and the other 12 months hence, how much ought to be paid 6 months hence to clear off both debts, interest being 4 p. c. per annum.

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85. The difference between two numbers is 375, and one of them is 7809; what is the other?

86. Simplify

$$\tfrac{1664}{1408} \text{ of } \{\tfrac{7}{10} \text{ of } £3\tfrac{3}{14} + 6\tfrac{2}{3} \text{ of } £3.\ 0s.\ 9d. - 4\tfrac{13}{21} \text{ of } £3.\ 2s.\}.$$

87. A fruit-seller has 1134 mangoes and 630 oranges. He forms them into heaps keeping the mangoes and oranges separate, and having the same number of fruits in each heap. If these heaps are as large as possible, how many fruits are there in each?

88. A cistern, the cubic content of which is 360 cu. ft., has two pipes which can empty it in 3 and 4 hours respectively. It has also a third pipe with an orifice of 1 sq. ft., through which water flows into the cistern at the rate of 1 yd. per minute. If all the three pipes be opened together when the cistern is full, in what time will it be emptied?

89. If 4 men or 6 women can do a piece of work in 20 days, in what time will 3 men and 2 women do it? On what supposition will the numerator of the fraction in your answer represent the number of *hours* they worked on the day to which the fraction refers?

90. Divide £1140 among *A*, *B*, *C*, in such a way that *A* may get half as much again as *B*, and *B* half as much again as *C*.

91. A dealer buys 10 horses at ₨400 each, 8 horses at ₨500 each and 4 horses at ₨600 each. He keeps the horses for 6 months, during which time each costs ₨15 a month, and sells them clearing $12\frac{1}{2}$ p. c. on his original outlay after paying all his expenses. Find the average selling price of each horse.

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92. A carriage and a horse are together worth ₨1200; if the carriage is worth ₨200 more than the horse, how much is the horse worth?

93. The population of a town is 60,000; if the births are 1 in 20, and the deaths 1 in 30 annually, what will the population become in one year?

94. A cistern, 9 ft. by 6 ft. by 5 ft., is emptied in 15 minutes by a pipe whose cross section is 36 sq. in.; how fast does the water flow in the pipe?

95. A race-course is $2\frac{1}{4}$ miles round. Four men start to walk round it. They walk at the rates of $3\frac{1}{4}$, $3\frac{3}{4}$, $4\frac{1}{2}$ and 5 miles per hour. How long will it be before they all meet again at the starting point?

96. 40 lb. troy of standard gold containing 11 parts in 12 of pure gold, is coined into 1869 sovereigns; calculate in grains the weight of pure gold in a sovereign.

97. Divide ₨7. 5*a.* into two parts, one of which is $\frac{5}{8}$ of the other.

98. If mangoes be bought at the rate of 13 for a rupee, how must they be sold to gain 30 per cent.?

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99. *A* has £324; *B* has £29 less than *A*; and *C*, if he had £205 more than what he has, would have as much as the double of *A* and *B* together: how much has *C*?

100. In how many years will the error amount to a day in considering the year to consist of $365\frac{1}{4}$ days instead of 365·242218?

101. The circumferences of two wheels measure 168 and 401 inches respectively; find the largest cogs which can be cut in each that they may work together.

102. The hands of a clock which gains uniformly at the rate of 15″ a day were set at sunset on the evening of the first of the month at 6 o'clock. The true time of sunrise on the 3rd was known to be a quarter to six, but the clock indicated a quarter past six. Find the error made in setting the clock on the 1st.

103. A train travels 30 miles an hour when it does *not* stop, and 25 miles an hour including stoppages; in what distance will the train lose one hour by stoppages?

104. Divide ₨123 among *A*, *B*, *C*, so that as often as *A* gets ₨3 *B* shall get $₨2\frac{1}{2}$, and as often as *B* gets ₨4 *C* shall get $₨3\frac{1}{2}$.

105. A merchant buys 4000 maunds of rice, $\frac{1}{2}$ of which he sells at a gain of 5 p. c., $\frac{1}{4}$ at a gain of 10 p. c., $\frac{1}{5}$ at a gain of 12 p. c., and the remainder at a gain of 16 p. c. If he had sold the whole at a gain of 11 p. c., he would have made ₨728 more. What was the cost of the rice per maund?

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106. A man sold 16 oranges to *A*, to *B* 4 more than to *A*, to *C* 5 less than to *B*; had he sold 3 less to each he would have left only one-third of what he had; find how many he had at first.

107. Simplify $\left\{\frac{1\frac{2}{3} \div 1\frac{3}{4}}{1\frac{4}{5} \div 1\frac{5}{6}} \div \frac{1\frac{6}{7} \div 1\frac{7}{8}}{1\frac{8}{9} \div 1\frac{9}{10}}\right\} \div \left\{\frac{\frac{1}{9} \div \frac{1}{10}}{\frac{1}{11} \div \frac{1}{12}} \div \frac{\frac{1}{13} \div \frac{1}{14}}{\frac{1}{15} \div \frac{1}{16}}\right\}$.

108. A room is 18 ft. long; and the cost of carpeting it is ₨72. If the breadth of the room were 4 ft. less, the cost would be ₨54; find the breadth of the room.

109. *A* can mow $2\frac{1}{2}$ acres of grass in $6\frac{2}{3}$ hours, and *B* $2\frac{1}{4}$ acres in $5\frac{1}{3}$ hours; in what time will they together mow a field of 10 acres, and how many acres will each mow?

110. The cost of 12 md. of wheat and 10 md. of gram is ₨50 when gram is at ₨2 per md. What is the price per md. of gram when 8 md. of rice and 6 md. of gram cost ₨34, the price of rice being $\frac{1}{4}$ higher than that of wheat?

111. Divide ₨20. 4*a*. among 5 persons so that the share of each (except the first) may be double of the shares of all who come before.

112. A merchant bought a 50-gallon cask of wine for ₨741. Supposing it to have lost 4 gallons, at what price per dozen bottles (nine bottles holding a gallon) should he sell it in order to gain 15 p. c. upon the whole original cost?

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113. A man lost as much by selling 20 chests of tea at ₨620 per chest as he gained by selling 25 chests at ₨692 per chest; what did each chest cost him?

114. A man left his property to two sons and a daughter; to the elder son he left $\frac{1}{4}$ of his property, to the younger son $\frac{1}{3}$, and to the daughter the rest, which was ₨4000 less than what the two sons together received: what was the entire property?

115. Three lines of paling run side by side for a distance of 864 yards. The rails are respectively 4, 6 and 9 feet apart. How often will a person walking outside the palings, on looking across them, see three rails in a line?

116. Three persons, *A*, *B*, and *C*, who can walk respectively 2, 3, and 4 miles per hour, start from the same place *P* at intervals of an hour. *A* starts first, and as soon as *B* has caught him up, *B* returns to the station *P*; find where he will meet *C*.

117. A fraudulent tradesman uses a yard measure one inch too short; what does he gain by his dishonesty in selling 20 yd. of cloth at ₨1. 2*a.* per yard?

118. *A*, *B*, *C* had each a cup of tea, containing 4 oz., 5 oz. and 6 oz. respectively. They blended their teas and then refilled their cups from the mixture; how much of the teas of *A* and *B* are contained in *C*'s cup?

119. If by selling wine at ₨6 per gallon I lose 25 per cent., at what price must I sell it to gain 25 p. c.?

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120. A man, having lived at the rate of £300 a year for 6 years, finds himself in debt, and reduces his expenditure to £250 a year; he is out of debt in 4 years: what is his income?

121. Express the sum of $\cdot\dot{5}7142\dot{8}$ of a viss, $\frac{3}{8}$ of $\frac{1}{3\frac{3}{8}}$ of $\frac{217}{384}$ of a maund and $\frac{3801}{10136}$ of a cwt. as a decimal of one ton. [One viss = 3 lb. 2 oz.; one maund = $82\frac{2}{7}$ lb.]

122. A rectangular cistern, 12 ft. long, 10 ft. wide and 4 ft. 3 in. deep, is filled with liquid which weighs 2040 lb. How much deep must another cistern be, which will hold 196 lb. of the same liquid, its length being 7 ft. and width 3 ft. 6 in.?

123. *A* can run 100 yd. in 12 sec., and *B* in 13 sec. How much start in distance must *A* give *B* in order that they may run a dead heat?

124. The Fort-Barracks are lighted with gas from 100 burners. Find the cost of lighting them per night of 10 hours, at the rate of ₨$5\frac{1}{2}$ for 1000 cu. ft. of gas, assuming that for the first 3 hours each burner consumes 1 cu. in. per second, and during the remainder of the night the light is so reduced that the consumption of gas by each burner is only $\frac{3}{4}$ of that quantity per second.

125. 120 coins consist of crowns, half-crowns and florins: the values of the crowns, half-crowns and florins are as 25 : 10 : 6: how many half-crowns are there?

126. A merchant sells 60 md. of rice at a profit of 8 p. c. and 94 md. at a profit of 10 p. c.; if he had sold the whole at a profit of 9 p. c. he would have received 17 annas less than he actually did: how much per md. did he pay for the rice?

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127. A man, having a certain number of mangoes to dispose of, sells half of what he has and one more to *A*, half of the remainder and one more to *B*, half of the remainder and one more to *C*, half of the remainder and one more to *D*; by which time he has only one left; find how many he had at first.

128. Simplify $\frac{3}{8}+\frac{5}{6}-\frac{2}{3}$ of $\frac{7\frac{1}{2}-5\frac{1}{4}}{1 \cdot 625}+\cdot 06\dot{4}74358\dot{9}$.

129. A dollar being worth 4*s.* 2*d.* and a rouble 3*s.* 1½*d.*, find the sum of money which can be paid by an exact number of either dollars or roubles, the number of roubles exceeding the number of dollars by 20.

130. *A* can do a piece of work in 15 days, *B* in 12 days and *C* in 10 days. All begin together; *A* leaves after 3 days, and *B* leaves 2 days before the work is done. How long did the work last?

131. A tank is 300 yd. long and 150 yd. broad; with what velocity per second must water flow into it through an aperture 2 ft. broad and 1½ ft. deep, that the level may be raised 1 ft. in 9 hours?

132. The height of the top of a flag-staff standing on a tower is 110 ft., and the height of the tower is 6 ft. more than 12 times the length of the flag-staff, what is the length of the flag-staff?

133. A merchant buys some cloth at such a price that by selling it at ₨4. 6*a.* per yd. he will gain 5 p. c. on his outlay. What percentage will he gain or lose if the cloth be sold at ₨3. 14*a* per yd.?

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134. I wish to buy an equal number of 3 kinds of toys, worth respectively 1*s.*, 1*s.* 6*d.* and 2*s.* 6*d.* each; how many can I get for £10?

135. In a book on Arithmetic an example was printed thus:

"Add together $\frac{1}{6\frac{2}{3}}$, $\frac{1}{5\frac{5}{6}}$, $\frac{1}{}$, $\frac{1}{8\frac{3}{4}}$,"

the denominator of one fraction being accidentally omitted. The answer given at the end of the book was $\frac{541}{840}$; required the missing denominator.

136. Find the side of a square courtyard, the expense of paving which at 3*s.* 9*d.* per sq. yd. was £42. 3*s.* 9*d.*

137. *A* and *B* start at the same time from Calcutta to Hugli and from Hugli to Calcutta respectively, each walking at the rate of 4 miles an hour. After meeting *B*, *A* increases his rate to 4½ miles an hour, and arrives at Hugli in 1⅓ hours from that time. After meeting *A*, *B* reduces his rate to 3½ miles an hour. In what time will he reach Calcutta?

138. If the rent of a farm of 24 acres be £39, what will be the rent of another farm of 36 acres, 5 acres of the former being worth 6 acres of the latter?

139. A purse contains £8 . 7 . 11, made up of pennies, shillings, half-crowns and crowns, the numbers of which are proportional

to 7, 3, 2 and 5 respectively; how many of each coin are there in the purse?

140. Calculate the profit per cent. made by a book-seller, assuming that he pays 11*s*. 4*d*. for a 16-shilling book and receives 25 copies for 24.

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141. A person mixes together 10 lb. of tea at ₨1. 4*a*. a lb., 12 lb. at ₨1. 6*a*., and 14 lb. at ₨1. 8*a*. He reserves 6 lb. of the mixture for himself and sells the remainder at ₨1. 13*a*. 4*p*. a lb. How much does he gain in money?

142. Multiply ·047321 by 12172·8144, using only 3 lines of multiplication.

143. Three men, the length of whose strides are 2 ft. 6 in., 3 ft. and 3 ft. 6 in., walk a mile. How often do they step together?

144. *A* and *B* start on a bicycle race. *A* has 10 minutes' start, during which he goes $2\frac{2}{3}$ miles; *B* rides at the rate of 16 miles an hour. Which will win in a race of 40 miles?

145. If 3 soldiers or 10 coolies can dig 150 cu. ft. of earth in 5 days, how many coolies must be employed to assist 7 soldiers in removing 580 cu. ft. of earth so as to get it done in 4 days?

146. 12*s*. $3\frac{1}{4}d$. is divided among men, women and children, whose numbers are proportional to 3, 5 and 7 respectively; if a man receives $5\frac{1}{2}d$., a woman $3\frac{1}{4}d$. and a child $2\frac{1}{3}d$., find the number of men.

147. An article was sold so as to gain 5 p. c. on its cost price. If it had been bought at 5 p. c. less, and sold for 1*s*. less, 10 p. c. would have been gained. Find the cost price.

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148. A wine merchant bought 7 gallons of wine at 17*s*. a gallon and 5 gallons at 15*s*. a gallon; he mixed the whole and added some water. The whole mixture he put into quart bottles which cost him 8*s*. 6*d*. and sold each bottle at 4*s*. and gained £1. 17*s*. 6*d*. on the whole. How much water did he mix?

149. Find the value of $\frac{15\frac{3}{5}}{7\frac{4}{5}}$ of £1 + $\frac{1}{5}$ of £140. 10*s*. 6*d*. ÷ $\frac{3}{5}$ of 21*s*.

150. The weight of water contained in a rectangular cistern, 8 ft. long, 7 ft. wide, is $93\frac{3}{4}$ cwt. Find the depth of water in the cistern, supposing a cu. ft. of water to weigh 1000 oz.

151. 25 men are employed to do a piece of work, who could finish it in 20 days; but the men drop off by 5 at the end of every 10 days: in what time will the work be finished?

152. If 48 men, working 8 hours a day for one week, can dig a trench 235 ft. long, 40 wide and 28 deep; in what time can

12 men, working 10 hours a day, form a railway cutting of 131,600 cu. yards? [A week=6 working days.]

153. The sum of areas of two circles, of which the diameters are as 3 is to 4, is equal to the area of another circle 10 ft. in diameter; find the diameters of the two circles, having given that areas of circles are to one another as the squares of their diameters.

154. A merchant sells sugar to a tradesman at a profit of 50 per cent.; but the tradesman becoming bankrupt pays only 5 annas in the rupee. How much per cent. does the merchant gain or lose by the sale?

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155. How many parcels of 6 lb. and 8 lb. each can a grocer make out of a hogshead of sugar, weighing 4 cwt. 3 qr. 14 lb., so as to have the same number of parcels of each sort?

156. *A* had 10*s.* in his purse, and *B* having paid *A* $2 \times \frac{3\frac{1}{7}}{1\frac{5}{6}}$ of £1. 11*s.* 6*d.* finds that he has remaining $\frac{1}{13}$ of the sum which *A* now has; what had *B* at first?

157. A number is exactly divisible by 11; but when divided by 5, 6 or 8 leaves always the remainder 1: find the least number which satisfies these conditions.

158. A boat's crew row over a course of $2\frac{1}{2}$ miles against a stream, which flows at the rate of 3 miles an hour, in 30 minutes. The usual rate of the stream is one mile an hour. Find the time which the boat would take in the usual state of the river.

159. If the cost of 11 miles of iron rails be R55000 when iron is selling at R95 a ton, what will be the cost of 19 miles of the same rails when iron is selling at R105 a ton?

160. A circular plate of gold, 10 in. in diameter and 2 in. thick, is melted and formed into two other circular plates, each 1 in. thick, whose diameters are as 3 to 4; find the diameters.

161. A man buys goods for R750, and sells $\frac{1}{4}$ of them at a loss of 4 p. c.; by what increase per cent. must he raise that selling price in order that by selling the rest at the increased rate he may gain 4 p. c. on the whole transaction?

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162. A person gives 53 guineas for 184 gallons of wine; how much water must he add to it, if he wishes to sell it at 5*s.* 3*d.* a gallon and make a profit of 7 half-guineas?

163. A vessel containing 21·84375 gallons of water is emptied by a pitcher which contains when full ·078125 gallon. How many times can the pitcher be filled entirely, and what fraction of a pint will it contain when the last quantity of water is poured into it?

164. A room is 8 yd. long ; the cost of carpeting it is ₨94. 8*a.*, and that of papering is ₨86. 10*a.* If the breadth of the room were 1 yd. more and its height 1 ft. less, the cost of carpeting would be ₨110. 4*a.* while the cost of papering would remain the same. Find the breadth and height of the room.

165. *A* and *B* run a race ; *A* has a start of 40 yd., and sets off 5 min. before *B*, at the rate of 10 miles an hour. How soon will *B* overtake him if his rate of running is 12 miles per hour ?

166. If the gas for 5 burners, lighted 5 hours every evening for 10 days, cost ₨3. 12*a.*, what will be the cost of 75 burners which are lighted 4 hours every evening for 15 days ?

167. Find the three highest integral numbers whose sum is under a thousand, so that the first may be $\frac{2}{3}$ of the second and second $\frac{4}{5}$ of the third.

168. A tradesman sells one kind of sugar at 3*a.* per seer and loses 20 p. c., and another kind at 5*a.* per seer and gains 25 p. c. He mixes the two together in equal proportions and sells the mixture at 6*a.* per seer. What is now his gain per cent. ?

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169. Two equal sums are divided, the one among 36 men, and the other among a certain number of women ; each man received ₨1. 4*a.* and each woman 10 annas less ; how many women were there ?

170. Simplify $\dfrac{\frac{4}{9} \text{ of } 1\frac{1}{2} - \frac{2}{3} \text{ of } \frac{5}{6}}{\frac{9}{100}(3\frac{4}{5} - \frac{1}{45} + 5\frac{1}{9})} \div \dfrac{6\frac{7}{8}}{4 + \dfrac{1}{\frac{5}{6} \text{ of } 2\frac{4}{15}}}$.

171. Three equal circular wheels revolve round a common horizontal axis ; the first makes a revolution in $5\frac{1}{3}$ minutes, the second in $2\frac{6}{7}$ minutes, and the third in $3\frac{3}{7}$ minutes. Three marks, one in each wheel, are in a horizontal line at a certain moment. What is the shortest interval after which they will be in a horizontal line again ?

172. *A* can do a piece of work in 6 hours, *B* in 8 hours and *C* in 10 hours ; how long will it take *C* to complete a piece of work, $\frac{1}{2}$ of which has been done by *A* working 7 hours and *B* working 8 hours ?

173. *A* walks $2\frac{1}{2}$ miles in 40 min., taking exactly a yard each step ; in what time will *B* walk $4\frac{5}{8}$ miles when his stride is 40 in. and he takes 21 steps while *A* takes 22 ?

174. Three persons, *A, B, C,* agree to pay their hotel bills in the proportion 4 : 5 : 6. *A* pays the first day's bill which amounts to £1. 5*s.* 5*d.* ; *B* the second which amounts to £1. 16*s.* 1*d.* ; and *C* the third which amounts to £1. 18*s.* 6*d.* ; how must they settle their accounts ?

175. A person bought a French watch bearing a duty of

25 per cent., and sold it at a loss of 5 per cent. ; had he sold it for £3 more, he would have cleared 1 per cent. on his bargain. What had the French maker for the watch ?

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176. An equal number of men, women and boys earn R165 in 6 days. If a woman earns 13*a.* 4*p.* a day, a man 8*a.* more, and a boy 8*a.* less, how many are there of each ?

177. What sum increased by $\frac{2}{3}$ of $\frac{7}{8}$ of $\frac{9}{20}$ of itself, amounts to £2463?

178. The length, width and depth of a cistern are 8 ft., 5 ft. 4 in. and 4 ft. 6 in. respectively. How many gallons does it contain, having given that a cu. ft. of water weighs 1000 oz. and that a pint of water weighs a pound and a quarter ?

179. *A* and *B* are termini of a railway 144 miles long. A fast train starts from *B* at 9 A. M. ; another fast train, travelling at the same rate, starts from *A* at 10 A. M. A slow train starts from *B* at 10-20 A. M. ; the fast train from *A* meets the other fast train at 11-30 A. M., and the slow train at 12-32 P. M. Find the rates at which the trains travelled.

180. If R1 = 1*s.* $10\frac{1}{2}$*d.*, £1 = 4·84 dollars, and 1 dollar = 5·2 francs, find the value in francs of 10 lacs of rupees.

181. Three merchants, *A*, *B*, *C*, trading with a capital of £3850, find after a certain time that their respective shares are increased by £66 . 7 . 6, £59 . 8 . 7 and £66 . 13 . 11 ; how much did *A* subscribe to the original capital ?

182. A grocer buys 200 lb. of tea, and sells 180 lb. for the same amount that he gave for the whole. The rest he sells at a profit of 20 per cent. What is his gain per cent. on the whole outlay ?

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183. The large wheel of an engine is 20 ft., and the small wheel 12 ft., in circumference. If the large wheel slips on an average 2 inches in every revolution, how many revolutions will the small wheel make more than the large one in going a distance of 12 mi. 1728 yd. ?

184. Calculate correctly to 7 places of decimals the value of

$$\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\frac{1}{7.9^7}+\ldots\ldots\ldots$$

185. The circumferences of the wheels of a carriage are $6\frac{1}{14}$ ft. and $8\frac{1}{18}$ ft. ; what is the least distance in which both the wheels will simultaneously complete an integral number of revolutions ? How often will the lowest points of the two wheels at starting touch the ground together in 10 miles ?

186. In a 200-yd. race *A* beats *B* by 20 yd., and *C* by 40 yd. By how many yards can *B* beat *C* in a 100-yd. race ?

187. On a piece of work 2 men and 5 boys are employed, who do $\frac{1}{2}$ of it in 6 days; after this 1 man and 1 boy more are put on, and $\frac{1}{3}$ more is done in 3 days: how many more men must now be put on if the work is to be completed in 1 day more?

188. A, B, C invest capital to the amount of £800, £600 and £500; A was to have $\frac{4}{9}$ of the profits which amount to £330; find C's share of the profits.

189. A tradesman defrauds his customers (i) by an adulteration of the article to the extent of 7 per cent., (ii) by using a balance which indicates 1 lb. when the amount in the other scale is really 15 oz. Which of the two practices is the more fraudulent, and to what extent is the customer cheated when he orders 1 lb. of the commodity?

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190. Find the distance between two towns when ₨309. 5*a.* 4*p.* is paid for the fare of 17 first class passengers at 1*a.* 8*p.* a mile, of 26 second class at 1*a.* 2*p.* a mile, and of 40 third class at 8*p.* a mile.

191. Find the value of $\left\{\frac{3\frac{1}{2} \text{ of } 5\frac{5}{8}}{2\frac{5}{6} \text{ of } 3\frac{1}{3}} \div \frac{2\frac{2}{11} \text{ of } \frac{11}{12}}{3\frac{1}{8} \text{ of } 7\frac{5}{7}}\right\}$ of $\frac{1s.\ 5d.}{4s.\ 7d.}$ of $\frac{2 \text{ ft. } 3 \text{ in.}}{5 \text{ ft. } 5 \text{ in.}}$ of 24 weeks 4 days 19 hours.

192. How many poles of fencing are required to enclose a square park containing 27 ac. 12 po. 1 yd.?

193. A, B, C can do a piece of work in 6, 8, 10 days respectively. They begin to work together; A continues to work till it is finished, B leaving off 2 days, and C 1 day before the work is completed. In what time is the work finished?

194. If the supply of a number of persons with bread at $7\frac{1}{2}d.$ the loaf for 31 days cost £27. 18*s.*; what will it cost to supply $\frac{2}{3}$ of that number for 20 days at $6\frac{1}{4}d.$ the loaf?

195. A, B, C purchase a farm for ₨10000, of which A pays ₨4000; they sell it so as to gain a certain sum, of which B takes ₨275 and C ₨175; find A's share of the profit.

196. One company guarantees to pay 5 per cent. on shares of 1000 rupees each; another guarantees to pay $4\frac{5}{6}$ per cent. on shares of 75 rupees each; the price of the former is 1245 rupees and of the latter 85 rupees. Compare the rates of interest which the shares return to the purchasers.

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197. If 5000 people took in hand to count a billion of sovereigns, and beginning their work at the commencement of the year 1852, could each count on the average 100 sovereigns in a minute (without intermission), when would they finish their task?

198. The total area of three estates is 1768 acres. If the areas of the two smaller estates be respectively three-fifths and two-thirds of that of the largest, find the acreage of each.

199. There are 3 pendulums, the first makes 35 beats in 36 seconds, the second 36 beats in 37 seconds, and the third 37 beats in 38 seconds. Supposing they commence together, find how many times they will again beat coincidently in 24 hours.

200. Sound travels at the rate of 1142 ft. per second; what is the distance of the thunder cloud, when the thunder succeeds the lightning at an interval of 9 seconds?

201. If 4 men and 6 women can do a piece of work in 5 days, which 5 men and 10 children can do in 4 days, or 3 women and 4 children can do in 10 days; find (i) how many men, (ii) how many women, (iii) how many children, could do the work in one day.

202. *A* and *B* enter into partnership; *A* puts into the business Rs.5000 more than *B*, who, as acting partner, is to have a salary of Rs.125 a month; at the end of 2 years the gross profits computed at $\frac{1}{6}$ of the capital per annum, are found to be Rs.7000, from which *B*'s salary is to be paid: find each one's share of the net profit.

203. The 3 per cents. are at $85\frac{1}{8}$; what price should the $3\frac{1}{2}$ per cents. bear, that an investment may be made with equal advantage in either stock? And what interest would be derived by so investing 5000*l*.

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204. Find the least sum of money that must be subtracted from £660. 7*s*. 4*d*. to make the remainder exactly divisible by 39.

205. What decimal must be added to

$$\frac{\frac{5}{2}(\cdot\dot{3}-\cdot 2)+\frac{1}{4}(\cdot\dot{3}+\cdot 2)}{\frac{1}{7}(2\cdot\dot{3}+\frac{\cdot 2}{11})+\frac{1}{11}(\frac{11}{5}-\frac{2}{7})} \text{ to produce unity?}$$

206. If gold can be beaten out so thin that one tola will form a leaf of 20 sq. yards, how many of these leaves will make up the thickness of a sheet of paper, the weight of a cu. inch of gold being $52\frac{11}{12}$ tolas and 432 sheets of the paper in thickness going to an inch?

207. A race-course is $\frac{1}{2}$ a mile long: *A* and *B* run a race and *A* wins by 10 yards; *C* and *D* run over the same course and *C* wins by 30 yards; *B* and *D* run over it and *B* wins by 20 yards; if *A* and *C* ran over it, which would win, and by how much?

208. Four men are employed to reap a field and after working 5 days they have cut 10 acres; 2 more men are then put on, and the whole is finished in 3 more days. How many acres are there in the field?

209. *A*, *B* and *C* are employed to do a piece of work for Rs.529;

A and *B* together are supposed to do $\frac{19}{23}$ of the work, and *B* and *C* together $\frac{8}{23}$ of the work: what should *A* be paid?

210. If ₹16430 be invested in the Govt. $4\frac{1}{4}$ per cent. loan at 106, what is the monthly income derived? Supposing that the loan is paid off at par in 10 years, what would be the rate of simple interest on the sum invested?

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211. 120 tons of coal are purchased for £87 . 16 . 9; find, to the nearest farthing, the price at which they must be retailed per ton so that no loss may be incurred; and at that price what profit will accrue?

212. Reduce to a decimal correct to 6 places:

$$\frac{1}{1\cdot3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\frac{1}{7\cdot3^7}+\dots\dots\dots$$

213. Find the greatest unit of time by means of which 11 hr. 31 min. 18 sec. and 23 hr. 4 min. $27\frac{1}{2}$ sec. can both be expressed as integers.

214. A man does $\frac{3}{5}$ of a piece of work in 18 days, and then gets a boy to help him. They work together for 3 days, when the boy leaves, and the man finishes the work in $7\frac{1}{2}$ days more. How long would it take the boy to do the whole?

215. If 10 horses and 98 sheep can be kept 9 days for £37 . 17 . 6; what sum will keep 45 horses and 216 sheep for 40 days, supposing 5 horses to eat as much as 76 sheep?

216. *A* starts business with ₹1200, and subsequently admits *B* who brings ₹1600. At the end of the year *A* receives $\frac{3}{5}$ of the profits; when was *B* admitted?

217. A man who has a certain capital calculates that if he invest it in $3\frac{1}{4}$ per cent. stock at 91, his income will be £25 more than if he invest it in 3 per cent. stock at 88. What is his capital?

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218. A tradesman buys 200 lb. of tea for £16, intending to gain one-fourth of his outlay by sale; but two pounds' worth at this calculation being damaged, at what price shall he sell the remainder per lb. to gain as much upon the whole outlay as he intended?

219. Express $(\frac{9}{10}+2\frac{1}{2})-(2\frac{3}{5}-1\frac{3}{4})\times\{(5\frac{1}{4}\times7\frac{2}{3})\div16\frac{1}{16}\}$ in its simplest form.

220. The diagonal of a square court-yard is 100 ft.: find the area.

221. Sound travels at the rate of 1140 feet a second. If a shot be fired from a ship moving at the rate of 10 miles an hour, how far will the ship have moved before the report is heard $14\frac{1}{4}$ miles off?

222. The length of the minute-hand of a church clock is $5\frac{1}{2}$ feet; what distance will the end of it travel through in 35 days, if 7 times the circumference of a circle be 22 times its diameter?

223. Three men A, B, C, undertake to complete in 20 days a piece of work for R247. 8*a*. A furnishes 10 men for 8 days and 6 men for the remaining days; B furnishes 7 men for 7 days and 12 men for 12 days; C furnishes 15 men who work on alternate days only until the work is completed. Find A's share of the sum.

224. A person having R8,500 in 4 per cent. Govt. bonds sells out when they are at $8\frac{1}{4}$ per cent. discount, and with the amount thus realised purchases 5 per cent. bonds which are at $6\frac{5}{8}$ per cent. premium: what does he gain or lose in annual income by the change?

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225. A contractor employs 100 men, 40 of whom work 10 hours on week days and only 5 hours on Sunday; the rest work 8 hours a day. If the wages of the former be 5*p*. per hour and of the latter 4*p*. per hour, what is the amount of wages paid in 4 weeks?

226. Two chests of tea of the same size and quality are consigned to A, B, C. A at first was to have $\frac{4}{5}$ of a chest, B [illegible], and C the rest. But A, B purchase $\frac{8}{11}$, $\frac{2}{11}$ of C's share respectively. How much will each have?

227. Find the side of the largest square tile, with which a court, 33 yd. 1 ft. 7 in. long and 20 yd. 11 in. broad, can be paved.

228. In a bicycle race of 2 miles over a circular course of 1 furlong, the winner in his last round overtook the second at a point in his 15th round. Their paces were as 159 to 149. At what distance was this point from the winning post?

229. If 3 men can do as much as 7 boys in a day, how many days will it take 25 boys to finish a piece of work of which 12 men have done a quarter in 13 days?

230. A, B, C hold a pasture in common for which they pay R16 per month; they put on it 70, 50 and 40 sheep respectively. A sells $\frac{2}{7}$ of his flock to B after 4 months, and after 3 months more C sells $\frac{2}{5}$ of his to A. How much of the rent should each pay at the end of the year?

231. A person bought 10 Bank of Madras shares at R1540 each and for 5 years got interest on his investment at the rate of $5\frac{1}{2}$ per cent. He then sold his shares at a loss of $22\frac{1}{2}$ per cent. How much did he make by the transaction, and what rate per cent. per annum had he for his money?

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232. A certain number of cows and twice as many sheep were bought for R94. 6*a*.; the cows cost R10. 3*a*. 6*p*. each and the sheep R4. 5*a*. 3*p*. each: how many sheep were bought?

233. The master of a ship, worth £5161. 3*s*. 9*d*., is himself owner of $\frac{3}{8}$ of $\frac{4}{5}$ of $\frac{2}{3}$ of her. He sells her for $\frac{5}{6}$ of her value; what is his own share?

234. The height of a square room is one-half of its breadth, and the cubic content of the room is 108 cu. yd.; find its dimensions

235. Two pipes, *A* and *B*, would fill a cistern in $37\frac{1}{2}$ min. and 45 min. respectively. Both pipes being opened, find when the second pipe must be turned off, that the cistern may be just filled in half an hour.

236. If 13 locomotive engines, each of 290 horse-power, working 11 hours a day for 7 days a week, can convey 7315 tons of goods to a distance of 221 miles in a given period, how many hours' work a day for 6 days a week must be done by 7 locomotives of 319 horse-power each, in order to convey 4845 tons of similar goods to a distance of 154 miles in an equal period?

237. How must teas at 2*s*. a lb. and 2*s*. 9*d* a lb. be mixed so that by selling the mixture at 2*s*. 8*d*. a lb. there may be a gain of 2*d*. per lb.?

238. If I sell 40 shares of ₨250 each in the Oriental Bank at 121 per cent. premium, how many shares of ₨1000 each in the Madras Bank at 72 per cent. premium can I buy, and how much will be left?

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239. Equal quantities of sugar, flour and rice were bought for ₨720. 9*a*.; the price of a md. of sugar is twice as much as that of a md. of flour, and the price of a md. of flour is twice as much as that of a md. of rice: find the cost of the sugar.

240. Find the value of $\frac{6\cdot757}{2\cdot17\dot{4}\dot{2}} \times \frac{\cdot\dot{2}5\dot{9}}{2\cdot78}$ of 12*s*. $9\frac{3}{4}$*d*.

241. A tea-merchant has a rectangular space for storing tea. It is $15\frac{3}{4}$ ft. long, $10\frac{1}{2}$ ft. broad and $9\frac{1}{3}$ ft. high. He wishes to fill this space with packets of a cubical shape, all of the same size. What is the largest size of such cubical packets that can be made to fill it exactly, and what would be the number of such packets?

242. A hare starts 40 yards before a greyhound and is not seen by him till she has been up 30 seconds. She runs at the rate of 12 and the hound at the rate of 15 miles an hour; how long will the chase last, and what distance will the hound have run?

243. If 3 men and 5 boys can reap 20 acres in 10 days, and if 5 men and 3 boys can reap 34 acres in 15 days, how many boys must assist 9 men, in order that they may reap 45 acres in 9 days?

244. A grocer bought 60 lb. of sugar of two different sorts for ₨16. 4*a*. The better sort cost 5*a*. per lb., and the worse 4*a*. per lb. Find how many pounds there were of each sort.

245. How much stock in the 3 per cents. must I sell to pay off a debt of £470, the price of the stock being $94\frac{1}{8}$, and the commission of $\frac{1}{8}$ on £100 of stock being also taken into consideration?

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246. How many four-anna pieces can be coined from 9 lb. of standard silver?

247. Find, by Practice, the dividend on a debt of £3471, at 13*s.* $7\frac{1}{2}$*d.* in the £.

248. The sides of a square are divided each into 8 equal parts, and lines are drawn through the points of division parallel to the sides. If the area of the square be 256 sq. ft., find the length of the side of each of the smaller squares, into which it is divided.

249. *A* and *B* run a mile race: at first *A* runs 5 yards to *B*'s 4, but after *A* has run half a mile he tires and runs 3 yards in the time in which he at first ran 5, *B* running at his original rate. Which wins, and by how much?

250. If the carriage of 150 ft. of wood, that weighs 3 stones per ft. cost ₨30 for 40 miles, how much will the carriage of 54 ft. of wood, that weighs 8 stones per ft., cost for 25 miles?

251. A greengrocer sells potatoes at 2*s.*, 2*s.* 6*d.* and 3*s.* 6*d.* a bushel, selling equal quantities of the first two kinds; what quantities of each kind does he sell, if the total quantity sold is 60 bushels, and if the average price obtained is 3*s.* a bushel?

252. A person invests 1250 gold mohurs in the Govt. five per cent. rupee stock at 105. The stock is converted subsequently to $4\frac{1}{2}$ per cents. at 95. Find the difference in his income, each gold mohur being considered equivalent to ₨17.

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253. If a person whose income is ₨1825 a year spend ₨44. 1*a.* a week for the first 20 weeks, to what must he limit his daily expenditure for the rest of the year so as not to be in debt at the end of it?

254. What number multiplied by itself will give $109\frac{025}{156}$?

255. A cubical block of marble whose edge is 2 ft. is placed within a rectangular cistern 4 ft. long, 3 ft. wide and 2 ft. deep, which is then filled with water; how many pounds of water must be taken out to reduce the surface 6 in.? [A cu. ft. of water weighs $62\frac{1}{2}$ lb.]

256. *A* and *B* can do a piece of work in $2\frac{2}{3}$ days, but when *B* works half time the work is done in 4 days. Show that *B* is twice as good a workman as *A*.

257. If 2 men and 5 women can do a piece of work in 8 days of 9 hours each; how long will it take 3 men and 6 women to do a piece of work twice as great, working 8 hours a day, the work of a man being double that of a woman?

258. Gold is 19 times as heavy as water, and copper 9 times. In what ratio should these metals be mixed that the mixture may be 15 times as heavy as water?

259. When the 3 per cents. were at 90 I found that by selling out and investing in the 4 per cents. at 95 I could improve my income by ₨243. What was the amount of my stock in the 3 per cents.?

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260. A person has in his drawer 15 piles of rupees, each containing 20: his servant steals them and puts in their place 15 piles, each consisting of 19 double-pice with a rupee at the top. How much does the person lose?

261. A person owes the sum of ₨31500, and ₨8500; and his property amounts to ₨14125 only. How much is he able to pay in the rupee; and what is the loss upon the second debt?

262. A rectangular piece of ground of 243 sq. yd. is one-third as broad as it is long; what is the distance round it?

263. A passenger train going 41 miles an hour, and 431 ft. long, overtakes a goods train on a parallel line of rails. The goods train is going 28 miles an hour, and is 713 ft. long. How long does the passenger train take in passing the other?

264. The distance by rail from Turin to Venice is 420 kilometres, and the first-class fare is 56 lire; find at the same rate in Indian money, the fare from Calcutta to Benares, a distance of 480 miles, reckoning 7 lire equal to ₨3 and 8 kilometres to 5 miles.

265. 40 lb. of coffee, at 2*s.* 6*d.* a lb., were mixed with a certain quantity of chicory at 1*s.* 9*d.* a lb., and the resulting mixture was worth 2*s.* a lb. How many pounds of chicory were there in the mixture?

266. How much money must be invested in the 3 per cent. consols when they are at $92\frac{1}{2}$, to produce the same income as would be produced by ₨1520 invested in the $3\frac{1}{2}$ per cents. at 95?

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267. If ₨20 . 7 . 6 be gained by selling an article for ₨79 . 10 . 9, how much would have been gained or lost by selling it for ₨59 . 7 . 6?

268. Find, by Practice, to the nearest penny, the rent of 375·3675 acres at £2. 19*s.* $10\frac{1}{2}$*d.* per acre.

269. Determine, by Duodecimals, the area of a rectangle whose adjacent sides are respectively 9 ft. $3\frac{1}{3}$ in. and 6 ft. $4\frac{1}{2}$ in.

270. *A* can beat *B* by 5 yd. in a 100-yd. race, and *B* can beat *C* by 10 yd. in a 200-yd. race; by how much can *A* beat *C* in a 400-yd. race?

271. If 210 coolies, in 7 days of 10 hours each, dig a channel,

1 mile long, 6 feet broad and 2 feet deep; in how many days of 7 hours each should 35 coolies dig a channel, 660 feet long, $7\frac{1}{2}$ feet broad and $2\frac{1}{4}$ feet deep? And how many cubic feet does each cooly dig in an hour?

272. The average of eleven results is 30; that of the first five is 25, and that of the last five is 28. Determine the sixth result.

273. What amount must be invested in the $4\frac{1}{2}$ per cent. stock at $103\frac{3}{4}$, in order to obtain, after deducting an income tax of $3\frac{1}{8}$ per cent., a clear income of ₨4000 a year?

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274. 4 thalers, 6 half-crowns and 8 florins amount to £2; what is the value of a thaler?

275. A reduction in the income-tax diminishes a tax, which is ₨15 when the tax is 8 pies in the rupee, by ₨3 . 12 . 0; what is the diminished rate of the tax?

276. The length of a room is twice its breadth and 4 times its height, and it contains 216 cu. yards of air; find its length.

277. *A* can reap a field in 5 days, and *B* in 6 days, each working 11 hours a day; in what time could they together reap it, working 10 hours a day?

278. If 38 men working 6 hours a day can do a piece of work in 12 days, find in what time 57 men working 8 hours a day can do a piece of work twice as great, supposing 2 men of the first set to do as much work in 1 hour as 3 men of the second set can do in $1\frac{1}{2}$ hours.

279. The average weight of 5 men is 5 st. 7 lb.; the average weight is diminished by 7 lb. when the weight of a boy is included: what is the weight of the boy?

280. A share-holder in a commercial company receives one year a dividend of 5 per cent. on his shares. The next year he receives a dividend of $7\frac{1}{2}$ per cent. and finds that he is ₨412. 8*a.* richer. Find the amount of his shares.

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281. To march at quick step is to take 108 paces of 2 ft. 8 in. per minute; what rate is this per hour?

282. A society subscribed ₨21. 5*a.* 4*p.* to a charity, each member paying as many pies as there were members in the society; find the number of members.

283. Find, by Duodecimals, the volume of a block of marble, 3 ft. 7 in. long, 2 ft. $3\frac{1}{2}$ in. wide and 1 ft. $2\frac{1}{4}$ in. deep.

284. A train, 880 feet long, overtook a man walking along the line at the rate of 4 miles an hour, and passed him in 30 seconds; the train reached the next station in 15 minutes after it had passed the man. In what time did the man reach the station?

285. If 40 men and 50 boys can do a piece of work in 6 days, working 6 hours a day, in how many days will 8 men and 20 boys do a piece of work half as large again, working 7 hours a day, assuming that a man does as much work in 3 hours as a boy in 5 hours?

286. The average age of 8 men is increased by 2 years, when one of them, whose age is 24 years, is replaced by a fresh man; what is the age of the new man?

287. If the price of the 4 per cents. just before the payment of a half-yearly dividend be 93, what ought to have been the price 3 months previously, supposing no change in the value of money to have taken place during that interval?

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288. The weekly wages at a mill amount to £186. 4*s.* In the mill a certain number of women are employed at 2*s.* 10*d.* a day, five times as many men at 5*s.* 6*d.* a day, and 6 times as many boys at 2*s.* 4*d.* a day: how many men are employed?

289. If the income-tax be 7*d.* in the £ in the first half of the year, and $3\frac{1}{2}d.$ in the second, what is the net income of a gentleman whose gross annual receipts are £1542 . 10 . 6?

290. An open cistern, made of sheet iron a quarter of an inch thick, is internally $62\frac{1}{2}$ in. long, 36 in. wide and 24 in. deep; find the weight of the cistern when full of water, if iron weighs 7 times as much as water and a cu. ft. of water weighs 1000 oz.

291. In a two-mile race *A* wins, *B* being 22 yd. behind, and *C* 106 yd. behind *B*. By how much would *B* beat *C* in a three mile race in which *A* does not run?

292. If the wages of 18 coolies for a month amount to R85 when rice is 24 seers per rupee, what ought the daily pay of a cooly be in proportion when the price of rice is R2. 10*a.* 8*p.* per maund?

293. *A* and *B* started on a race and ran a distance exactly together. Then *B* began to fail and gave up the race when he had run 56 yards farther, *A* having gone during the same time 320 yards. The average of the entire distances run by the two men was 1188 yards. What distance had they run together?

294. The £23 shares of one company pay a dividend of £1 per share; the £15 shares of another yield £·725 per share. The market value of the former is £24·92, of the latter £17. Compare the rates of interest returned to the purchasers.

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295. A man bought 100 oranges at 2 a pice, and 100 more at 3 a pice, and mixed and sold the whole at 5 for 2 pice; how much did he lose?

296. Find, by Practice, the cost of fencing 3 mi. 3 fur. 180 yd. 1 ft. 6 in. of road at £479. 15*s.* per mile.

297. An open cistern, made of sheet iron $\frac{1}{2}$ inch thick, is externally 10 in. long, 8 in. broad and $5\frac{1}{2}$ in. deep; find the price of the cistern at ₨8 per cwt., if a cu. ft. of iron weighs $4\frac{1}{2}$ cwt.

298. *A* does half as much work again as *B* in the same time, and *B* does one-third as much again as *C*; working together they can do a certain work in 5 days; but if after working 2 days *A* leaves off, how long will *B* and *C* take to finish it?

299. When rice is 10 seers the rupee, 7 persons can be fed for 30 days at a certain cost. For how many days can 6 persons be fed at the same cost when rice is 14 seers the rupee?

300. If the daily wages of a labourer rise from 4*a.* 9*p.* to 6*a.*, what percentage of the increase in the price of food and other commodities will cause his position to be unaltered?

301. A person buys 5 shares in a company, and sells three of them at a gain of 10 per cent. and the remaining two at a gain of $16\frac{2}{3}$ per cent. The gain on the latter sale is £2 . 19 . $7\frac{1}{2}$ more than on the former. How much did he pay for each share?

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302. A man buys 25 seers of milk at 1*a.* 6*p.* a seer, and sells it at 1*a.* 3*p.* a seer, making a profit of 5 annas; how many seers of water did he add to the milk?

303. Now that the income-tax is 5 pies in the rupee, a person's net income is ₨374 per mensem; what will it be when the income-tax is raised to 7 pies?

304. Find, by Duodecimals, the area of a square whose side is 12 ft. 8 in. 4 pt.

305. A train starts from *A* at 12 o'clock and runs towards *C*, which is 100 miles distant, at the rate of 30 miles an hour; at the same time the mail coach starts for *C*, from *B*, which is half way between *A* and *C*, and runs at 10 miles an hour; at what distance from *C* will it be overtaken by the train?

306. If 13 solid inches of copper balance 17 of iron, and 15 of iron balance 16 of tin, and 19 of tin balance 12 of zinc, how many solid inches of zinc balance 2470 solid inches of copper?

307. If the income-tax be 6 pies in the rupee for the first half of the year and 3 per cent. in the second, what is the gross income of a gentleman whose net annual receipts amount to ₨1454. 1*a.*?

308. What sum must a person invest in the 3 per cents. at 90, in order that by selling out £1000 stock when they have risen to $93\frac{1}{2}$, and the remainder when they have fallen to $84\frac{1}{4}$, and investing the whole proceeds in the 4 per cents. at par he may increase his annual income by £9. 5*s.*?

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309. Divide ₨115. 2*a.* among 20 boys and 25 girls, so that each boy may receive 12 annas more than each girl; how much will each boy receive?

310. Three-fifths of the square of a certain number is 126·15; what is the number?

311. An open cistern whose capacity is 4320 gallons is externally 14·1137 ft. long, 10·25 ft. wide and 5·1$\dot{6}$ ft. deep; the sides are $1\frac{1}{2}$ in. thick; find the thickness of the bottom, having given that a gallon contains 277·274 cu. inches.

312. *A* and *B* walk a race of 10 miles; *A* gives *B* 20 minutes' start; *A* walks uniformly a mile in $17\frac{1}{2}$ minutes and catches *B* at the 8th mile-stone; find by how much *B* lost in time and space.

313. If 17 men can build a wall 100 yd. long, 12 ft. high and $2\frac{1}{2}$ ft. thick, in 25 days, how many men will build a wall twice the size in half the time?

314. In 1861 three towns had populations of 17650, 19600, 18760, respectively. In 1871 the population of the first had decreased 18 per cent., that of the second had increased 21 per cent., while the population of the third had increased by 4690; find the change per cent. in the total population of the three towns.

315. A gentleman invests ₨5600 in the $5\frac{1}{2}$ per cent. Govt, paper, and derives therefrom an annual income of ₨275. At what premium was the $5\frac{1}{2}$ per cent. paper at the time he invested?

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316. Find the circumference of the wheel of a locomotive, which makes 5 revolutions in a second, and which performs a journey of 30 miles in 44 minutes.

317. A man has an income of £200 a year; an income-tax is established of 7*d.* in the £, while a duty of $1\frac{1}{2}d.$ per lb. is taken off sugar; what must be his yearly consumption of sugar that he may just save his income-tax?

318. *A*, *B*, *C* are three spouts attached to a cistern. *A* can fill it in 20 min., *B* in 30, and *C* can empty it in 40 min. If *A*, *B* and *C* be opened successively for one minute each, in what time will the cistern be filled?

319. A besieged garrison consists of 300 men, 120 women and 40 children, and has provisions enough for 200 men for 30 days. If a woman eats $\frac{2}{3}$ as much, and a child $\frac{1}{2}$ as much, as a man, and if after 6 days 100 men with all the women and children escape, how long will the remaining provisions last the garrison?

320. The price of rice being raised 50 per cent., by how much per cent. must a house-holder reduce his consumption of that article so as not to increase his expenditure?

321. The owner of 4 per cent. Govt. paper, bringing in ₹8976 per annum, exchanges it for 5 per cent. paper. His annual interest is increased by ₹44. What is the increase or decrease of his nominal capital?

322. A bill on London for £175 drawn at 6 months after sight, is purchased at Madras, the rate of exchange being 2*s*. $0\frac{5}{8}$*d*. the rupee. Four months before it becomes due, it is discounted in London at the rate of $2\frac{1}{2}$ per cent. (per annum) discount. What was paid for the bill in Madras, and what does it realise in London?

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323. A man laid out £30. 15*s*. in spirits which he bought at 15*s*. a gallon; he retailed them at 17*s*. 6*d*. a gallon, making a profit of £4. 5*s*.: how many gallons must he have lost by leakage?

324. Arrange $\sqrt{2}$, $\sqrt[3]{3}$ and $\frac{4}{3}$ in order of magnitude.

325. Two trains, running at the rates of 25 and 20 miles an hour respectively on parallel rails in opposite directions, are observed to pass each other in 8 seconds, and when they are running in the same direction at the same rates as before, a person sitting in the faster train observes that he passes the other in $31\frac{1}{2}$ seconds; find the lengths of the trains.

326. If 6 dollars and 6 roubles are together worth £1. 13*s*. 9*d*., and 4 dollars and 8 roubles are together worth £1. 11*s*. 8*d*., what is the value of 6 dollars and 8 roubles?

327. In an examination *A* obtains 10 per cent. less than the minimum number of marks required for passing; *B* obtains $11\frac{1}{9}$ per cent. less than *A*; and *C* $41\frac{3}{17}$ per cent. less than the number of marks obtained by *A* and *B* together. Does *C* pass or fail?

328. I have ₹6500 to invest in public securities. Will it be most to my advantage to invest it in the 5 p. c. Govt. loan which is at $10\frac{3}{4}$ per cent. discount, or to purchase at par Treasury Bills which bear an interest of 3 pies per cent. per diem? Calculate the difference.

329. If the par of exchange be two English shillings for the Indian rupee, but if an Indian bill of exchange for ₹540. 12*a*. be negotiated in London for £51. 10*s*., how much per cent. below par is the rate of exchange?

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330. On Monday January 3, 1888, a man commenced to subscribe for a daily pice paper (published on week days only); what had he spent by June 13th of the same year?

331. A gentleman's income is diminished by £150; but the income-tax being raised from 6*d*. to 7*d*. in the £., he pays the same amount of tax as before; find his present income.

332. *A* and *B* start to run a race; their speeds are as 17 to 18,

A runs $2\frac{1}{3}$ miles in 16 min. 41 sec.; *B* finishes the course in 34 min.: determine the length of the course.

333. If 5 men and 8 boys reap 9 acres in 10 days, and 4 men and 4 boys reap 3 acres in 5 days, how many acres will 2 men and 3 boys reap in 7 days?

334. To 432 gallons of a mixture of brandy and rum, which contains $8\frac{3}{4}$ per cent. of brandy, some water is added, and the proportion of brandy in the mixture is thereby diminished to $7\frac{1}{2}$ per cent. How much water is added?

335. A person who has £1900 Russian 4 per cent. stock sells out at 104 and devotes £962. 13*s*. 4*d*. to the purchase of 3 p. c. consols at 95, and lends the rest of the sum realised on mortgage. What interest must he ask for his money that his income may be the same as before?

336. If the rate of interest for money be 3 per cent., what should be the rate of exchange for bills payable at sight in England when the rate for those payable 4 months after sight is 1*s*. $8\frac{1}{8}$*d*. per rupee?

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337. A merchant buys 60 yards of cloth; he sells half of it at a gain of 3 annas per yard, and the remainder at a gain of 2 annas per yard, and realises R44. 1*a*. What was the cost price per yard?

338. A man buys a number of mangoes for R9, the price in pies of each mango being equal to the square root of the number purchased; find the number purchased and the price of each.

339. A train which travels at the uniform rate of 30·8 ft. a second, leaves Madras at 7 A. M.; at what distance from Madras will it meet a train which leaves Arconum for Madras at 7-20 A.M., and travels one-third faster than it does, the distance from Madras to Arconum being 42 miles?

340. If 5 men, 2 women and 3 boys, or 6 men and 4 boys, can mow 3 acres in 5 days; how many acres would 3 men, 2 women and one boy mow in 11 days, supposing a man to do as much work as 3 boys?

341. A person loses in his first year 23 per cent. of his capital, but in the next year he gains 40 per cent. of what he had at the end of the first year, and his capital is now R720 more than it was at first; find his original capital.

342. A person invested equal sums of money in the 3 per cents. at $97\frac{1}{2}$, and in the $3\frac{1}{2}$ per cents. at $102\frac{1}{2}$; his resulting income was £259. 10*s*. How much did he invest?

343. A merchant in London receives two bills, drawn at 4 months after sight, each for R5000; one he discounts immediately,

the rate of interest being 3 per cent. per annum; the other he keeps till maturity, and then exchanges at the rate of 1s. 9d. per rupee and finds that he has got as much as he did for the first bill. What was the rate of exchange when the first bill was discounted?

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344. A man, having bought 128 yards of cloth for R80, sells one-fourth at a loss of 2 annas per yard; by how much must he raise that selling price, in order that, by selling the rest at the increased rate, he may gain 2 annas per yard on the whole?

345. Incomes below £150 a year being subject to 5*d.* in the £ income-tax, and incomes above £150 to 7*d.* in the £; find what income above £150 a man must have, that he may be just 7¾*d.* a year poorer than a man who has £149. 10*s.* a year.

346. *A* and *B* run a mile, and *A* wins by 160 yd.; *A* and *C* run over the same course and *A* wins by 20 min.; *B* and *C* run over it and *B* wins by 12 min. In what time can *A* run a mile?

347. If 16 darics make 17 guineas, 19 guineas make 24 pistoles, 31 pistoles make 38 sequins, then how many sequins are there in 1581 darics?

348. What sum must be paid on the insurance of a cargo of the value of R33575. 4*a.* so that in case of loss the cargo and all expenses of insurance may be recovered? The premium is at the rate of 4·725 per cent., policy duty 3½ annas per cent. and agent's commission ½ per cent.

349. A person has £26041 of a 4 per cent. stock. He saves each year ¼ of his income, which he invests at 4 per cent. What is his income in the 4th year?

350. If gold be at a premium of 5 per cent., and a person buy goods marked 300 rupees, and offer gold to the amount of 300 rupees, what change ought he to receive in notes, 5 per cent. being abated for ready payment?

PROBLEMS. 175.

1. By what number less than 1000 must 4389 be multiplied so that the last three figures (to the right) of the product may be 438?

2. If 5 cwt. 3 qr. 14 lb. cost £6 per cwt., what will be the cost per pound when the cost of the whole has been reduced by £7. 16*s.* 8*d.*

3. On measuring a distance of 32 yards with a rod of a certain length it was found that the rod was contained 41 times with half an inch over; how many inches will there be over in measuring 44 yards with the same rod?

4. Find the least number above 1000, which when divided by 5 or by 6 or by 9, will leave the same remainder 3.

5. A bill of £100 was paid with guineas and half-crowns, and 48 more half-crowns than guineas were used; find how many of each were paid.

6. *A* has twice as much money as *B*. They play together, and at the end of the first game *B* wins from *A* one-third of *A*'s money; what fraction of the sum which *B* now has must *A* win back in the second game that they may have exactly equal sums?

7. What is the smallest whole number which is exactly divisible by $1\frac{5}{28}$, $2\frac{2}{21}$ and $3\frac{1}{7}$?

8. *A* pays £9 . 3 . 4 more rates than *B*, their incomes being equal: living in different towns they are rated at 2*s*. and 1*s*. 4*d*. in the £ respectively; what is their income?

9. A pint of water weighs a pound and a quarter, and a cu. foot weighs 1,000 oz.; how many gallons are there in a cu. foot? How many gallons will fill a cistern 5 ft. long, $2\frac{1}{2}$ feet wide and 2 ft. deep?

10. A gallon contains 277·274 cu. in.; a cu. ft. of water weighs 1000 oz. How many gallons weigh a ton? and what is the weight of a pint?

11. If 162 gallons fill a cistern $5\frac{1}{3}$ ft. by $4\frac{1}{2}$ ft. by $1\frac{1}{8}$ ft., find the number of cu. inches in a pint.

12. If a cu. inch of water weighs 252·45 grains, which is the more accurate of the following rough statements:—a cu. ft. of water weighs 1000 oz., a cu. yd. weighs $\frac{3}{4}$ of a ton?

13. If a decilitre be ·052 gallon, find the value of a pint of liquid which is worth 2 francs the decilitre: 1200 francs being equal to £49.

14. Three men are employed on a work, working respectively 8, 9, 10 hours per day, and receiving the same daily wages. After three days each works one hour a day more, and the work is finished in three days more. If the total sum paid for wages be £2 . 7 . $6\frac{1}{4}$, how much of it should each receive?

15. The sum of two numbers is 5760, and their difference is equal to one-third of the greater: find the numbers.

16. Two casks contain equal quantities of beer; from the first 34 quarts are drawn, and from the second 80; the quantity remaining in one cask is twice that in the other. How much did each cask originally contain?

17. Shew that if the price in rupees of a cwt. of goods is divided by 7, the result is the price in annas of a lb. weight of the goods.

18. If ₨72 be divided among 5 men, 7 women and 13 boys so that 2 men receive as much as 5 boys, and 2 women as much as 3 boys, how much will each man, woman and boy receive?

19. How many revolutions will be made by a wheel which revolves at the rate of 329 revolutions in 3 min. while another wheel revolving 431 times in 4 min. makes 2586 revolutions?

20. If a train goes $22\frac{1}{2}$ miles an hour, how many revolutions does the driving-wheel, 11 ft. in circumference, make in a second?

21. A game licence costs 15*s.*, and a cartridge 2*d.* A sportsman kills his bird once in 5 shots. If birds are worth 2*s.* 6*d.* a brace, how many birds must be shot just to pay expenses?

22. A vulgar fraction has for its numerator 157, and its nearest approximate value in thousandths is ·370; what is the denominator?

23. A man after a tour in England finds that he had spent every day half as many rupees as the total number of days he had been from home. His tour cost ₨1800. How many days did it occupy?

24. A plate of metal is beaten to the thickness of $\frac{1}{8}$ of an inch and the weight of a circular medal cut from it, whose diameter is $1\frac{1}{6}$ inches, is $1\frac{1}{4}$ oz. troy. If the same plate be beaten to the thickness of $\frac{1}{9}$ of an inch, what will be the weight of a medal cut out of it of the diameter of $1\frac{1}{7}$ inches (the areas of circles being proportional to the squares of their diameters)?

25. It is said that 240,000 letters are posted in Berlin daily 16·6 per cent. of which are town letters. This gives one letter for every 3 persons in Berlin; what is its population?

26. The French unit of linear measure is a *metre* equal to 39·371 English inches; the square formed on a line of 10 metres (called an *are*) is the French unit of surface. Find the equivalent in English square measure, of a hectare (100 ares).

27. A rectangular swimming bath is 60 ft. long and 40 ft. broad; it can be filled by a supply-pipe in 5 days, and if 6,000 cubic feet of water be thrown in, the rest can be filled in 3 days 18 hours. Find the depth of the bath.

28. The debts of a bankrupt amount to ₨21345. 4*a.* and his assets consist of property worth ₨9167. 10*a.* 8*p.* and an undiscounted bill of ₨5130 due 4 months hence, simple interest being reckoned at 4 p. c. per annum. How much in the rupee can he pay his creditors?

29. The diameter of the fore-wheel of a carriage is $1\frac{3}{4}$ ft. and that of the hind-wheel is 3 feet; how far will the carriage have travelled when the fore-wheel has made 100 more revolutions than the hind-wheel? (The circumference of a circle : diameter :: 3·1416 : 1.)

30. Tea at 4*s.* $3\frac{1}{2}$*d.* per lb. is mixed with tea at 3*s.* $7\frac{1}{2}$*d.* per lb. so that the mixture contains 72 per cent. of the former. Find the weight of a chest of this mixture which is worth £6. 16*s.* 10*d.*

31. A merchant buys China tea at 3*s.* 6*d.* per lb. To improve the flavour he adds 2 oz. of Assam tea to every lb. of China tea, and finds that the mixture costs him 4*s.* per lb. How much per lb. did he give for the Assam ?

32. Standard silver, of which 111 parts in 120 are pure silver being worth ₨31 per lb., find the value of a Sicca Rupee which weighs 7 dwt. 12 gr. and has a fineness of 979 parts in 1000.

33. A contract is to be finished in 5 months and 17 days, and 43 men are put on to work at once ; at the end of $\frac{2}{3}$ of the time it is found that only $\frac{1}{7}$ of the work is done ; what extra number of hands will be required to complete the contract in the given time, the last employed men to work 12 hours a day, whilst the first 43 men work until the contract is completed only 10 hours a day ?

34. A man can do as much work in 4 hours as a woman in 6 hours, or as a boy in 9 hours ; how long will it take a boy to complete a piece of work, one-half of which has been done by a man working 10 hours and a woman working 16 hours ?

35. If a piece of cloth, 4 yd. long and 15 in. wide, cost ₨3. 2*a.*, how much should you give for another piece, 19 yd. long and 12 in. wide, every sq. in. of which is worth $\frac{2}{3}$ of the value of a sq. ft. of the former ?

36. A person sets out to walk 26 miles ; for a quarter of the distance he goes at the rate of 5 miles an hour, for half the remaining distance at 4 miles an hour and 3 miles an hour for the other half. State the exact time occupied in the journey.

37. How often between 12 and 1 are the hands of a clock an integral number of minute-spaces apart ?

38. Two clocks begin striking the hour of noon together on a certain day, the interval between every two strokes being 1″ and 2″ respectively. They gain 1″ and 2″ respectively in every 24 hours. After what length of time will they end striking the hour of noon together ?

39. *A* and *B* start at the same time on a journey. *A* walks at the rate of 4 miles an hour, and *B* of 3 miles an hour. When *A* has gone half way, *B* gets a ride and goes at twice the rate of *A*, until he has ridden a distance equal to $\frac{2}{13}$ of the whole journey beyond the spot at which he passes *A*. *B* then walks the remainder of the journey, *A* having walked it all. Will *A* or *B* arrive first ? And what fraction of the whole journey will the other still have to travel ?

40. If 15 men can dig 600 cu. ft. of earth in 5 days, working

8 hours a day, how many men would be required to dig 1575 cu. ft. in 14 days, working 9 hours a day, supposing that a man who works 8 hours a day does in 25 hours the same amount of work that a man who works 9 hours does in 26 ?

41. If 21 horses and 217 sheep can be kept 10 days for the same sum as it would cost to keep 9 horses and 60 sheep for 27 days, find how many sheep eat as much as 3 horses.

42. In running a four-mile race on a course half a mile round, *A* overlaps *B* at the middle of the 6th round. By what distance will *A* win ?

43. *A* and *B* start to run a race at 3 o'clock. The winner comes in at $6\frac{3}{4}$ minutes past 3, beating the other by 40 yards. At 4 minutes past 3 the loser was 1140 yards from the winning-post. Find the length of the course, and the speed of the winner in miles per hour.

44. Five men do $\cdot 600\dot{6}$ of a piece of work in 2·12 hours, how long will 6 boys take to finish it, it being known that 3 men and 7 boys have done the whole of a similar piece of work in 3 hours ?

45. If 4 men earn as much in a day as 7 women, and one woman as much as 2 boys, and if 6 men, 10 women and 14 boys working together for 8 days earn £22, what will be the earnings of 8 men and 6 women working together for 10 days ?

46. The distance by Railway from Madras to Salem is $206\frac{3}{4}$ miles. A Passenger Train travelling 20 miles an hour leaves Madras at 7 A. M. ; and a Special Train at 10 A. M. the same day. At what rate must the latter travel, so as just to overtake the former at Jollarpett Junction (132 miles from Madras), and at what hour must a Goods Train leave Salem for Madras travelling 15 miles an hour, so as to reach Jollarpett at the same time as the other Trains ?

47. Two trains measuring 330 ft. and 264 ft. respectively, run on parallel lines of rail. When travelling in opposite directions they are observed to pass each other in 9 seconds, but when they are running in the same direction at the same rates as before the faster train passes the other in $27\frac{1}{2}$ seconds. Find the speeds of the two trains in miles per hour.

48. A man near the sea-shore sees the flash of a gun fired from a vessel, steaming directly towards him, and hears the report in 15″. He then walks towards the ship at the rate of 3 miles an hour, and sees a second flash 5 minutes after the first, and immediately stops ; the report follows in 10·5″. Find the rate of the ship, the velocity of sound being 1200 feet per second.

49. A soldier has 4 hours' leave of absence ; how far may he ride on a coach which travels 8 miles an hour, so as to return to the camp in time, walking at the rate of 4 miles an hour ?

50. Two trains start at the same time, the one from Calcutta to Allahabad, the other from Allahabad to Calcutta. If they arrive at Allahabad and Calcutta respectively 5 hours and 20 hours after they passed each other, show that one travels twice as fast as the other.

51. A cistern is provided with two pipes, A and B. A can fill it in 20 minutes, and B can empty it in 30 minutes. If A and B be kept open alternately for one minute each, how soon will the cistern be filled?

52. A, B, C are pipes attached to a cistern. A and B can fill the cistern in 20 and 30 minutes respectively, while C can empty it in 15 minutes. If A, B, C be kept open successively for one minute each, how soon will the cistern be filled?

53. A train having to perform a journey of 150 miles, is obliged after 100 miles to reduce its speed by one-fifth. The result is that the train arrives at its destination half an hour behind time. What is its ordinary rate?

54. A down Passenger Train, 176 yd. long, travelling at the rate of 20 miles an hour, meets at 7 A. M. an up Goods Train, $293\frac{1}{3}$ yd. long, and passes it in 24 seconds. At 7-30 A. M. the down Passenger meets the up Mail, 88 yd. long, and passes it in 12 seconds. When will the Mail overtake the Goods?

55. A and B start together from the same point on a walking match round a circular course. After half an hour A has walked 3 complete circuits, and B four and a half. Assuming that each walks with uniform speed, find when B next overtakes A.

56. A certain sum is to be divided among A, B and C. A is to have £30 less than the half, B is to have £10 less than the third part, and C is to have £8 more than the fourth part. What does each get?

57. £4212 is divided among A, B, C, so that A receives $\frac{4}{9}$ as much as B and C together, and B $\frac{4}{5}$ of what A and C together receive. Find how much each receives.

58. Two-thirds of a certain number of persons received 18*d.* each, and one-third received 2*s.* 6*d.* each. The whole sum spent was £2. 15*s.* How many persons were there?

59. A crew which can pull at the rate of 9 miles an hour, finds that it takes twice as long to come up a river as to go down: at what number of miles an hour does the river flow?

60. A, B, C are partners: A whose money has been in the business for 4 months claims $\frac{1}{8}$ of the profit: B whose money has been in the business for 6 months claims $\frac{1}{3}$ of the profits: C had ₨1560 in the business for 8 months: how much money did A and B contribute to the business?

61. Two persons A and B rent a field. A puts on it 12 horses for $2\frac{1}{2}$ months, 20 cows for 4 months and 50 sheep for 5 months; B puts 18 horses for $3\frac{1}{2}$ months, 15 cows for 5 months and 40 sheep for $4\frac{1}{2}$ months. If in one day 3 horses eat as much as 5 cows, and 6 cows as much as 10 sheep, what part of the rent should A pay?

62. A can dig a trench in $\frac{1}{2}$ the time that B can; B can dig it in $\frac{2}{3}$ of the time that C can; all together they can dig it in 6 days. Find the time it would take each of them alone.

63. For 5 guineas can be obtained either 12 lb. of tea and 15 lb. of coffee, or 36 lb. of tea and 9 lb. of coffee; find the price of a pound of each.

64. Divide 48 into two parts such that if one part be multiplied by 3 and the other by 5, the sum of the products shall be 180.

65. Divide 20 into two parts such that three times one part may be equal to twice the other part.

66. A decimetre is equal to 3·937 inches, and a cubic decimetre of water weighs 1 kilogram. If a cubic inch of water weighs 252·45 grains, express a kilogram in pounds avoir. correct to two decimal places.

67. Twenty gallons of liquid contain 60 per cent. of nitric acid and the rest water. How many gallons of water should be added to the mixture to lower the proportion of nitric acid to 40 per cent.?

68. Divide R1000 among 1 man, 3 women and 36 children so that the man gets 4 times as much as each woman, and the women together get 12 times as much as each child.

69. Two men undertake to do a piece of work for R40. One could do it alone in 5 days, the other in 8 days. With the help of a boy they finish it in 3 days. How should the money be divided?

70. The sum of the ages of A and B is now 55 years, and their ages 10 years ago were as 4 is to 3; find the present ages.

71. A tradesman's prices are 20 p. c. above cost price; what profit does he make, if he allows his customers a discount of a penny in the shilling?

72. Four apples are worth as much as 5 plums, 3 pears as much as 7 apples, 8 apricots as much as 15 pears, and 5 apples sell for 2*d.* I wish to buy an equal number of each of the four fruits, and to spend an exact number of pence: find the least sum I can spend.

73. The manufacturer of an article makes a profit of 20 per cent., the whole-sale dealer, of 10 per cent., and the retail-dealer, of 5 per cent. What is the cost of the manufacture of an article which is retailed for R7. 8*a.* 9*p.*?

74. Two cogged wheels, of which one has 16 cogs and the other 20, work in each other. If the latter turns 60 times in $\frac{2}{3}$ of a minute, how often does the former turn in 16 seconds ?

75. The price of butter having risen 25 p. c., the daily allowance of each person in a family is reduced from 1 oz. to $\frac{4}{5}$ oz. If the monthly charge for butter is thenceforward 12*s*., what was it before the changes were made ?

76. A bankrupt has book-debts equal in amount to his liabilities, but on £4000 of them he can recover only 15*s*. in the £, and the expenses of the bankruptcy are £200 ; if he pay 15*s*. $2\frac{2}{5}$*d*. in the £, what is the amount of his liabilities ?

77. A ship 40 miles from the shore springs a leak which admits $3\frac{3}{4}$ tons of water in 12 minutes. 60 tons would suffice to sink her, but the ship's pumps can throw out 12 tons of water in an hour. Find the average rate of sailing so that she may reach the shore just as she begins to sink.

78. Standard silver is formed by mixing 11 parts of fine silver with one of copper. How many rupees can be coined from 1 lb. avoir. of fine silver, if 1 lb. troy of standard silver is coined into 32 rupees ?

79. If $2\frac{1}{4}$ tolas of gold, 22 carats fine, be worth ₨49. 8*a*., of what fineness must gold be in order that $1\frac{1}{2}$ tolas of it may be worth ₨34. 8*a*. ?

80. A man having to walk 36 miles finds that in 3 hr. 20 min. he has walked $\frac{5}{7}$ of the remaining distance ; find his speed.

81. Supposing the alloy in a rupee to be $\frac{1}{12}$ of the mass, and the coin to be worth 2 pice if it were all alloy, what would be its exact value if it were all pure silver ?

82. A mixture contains wine and water in the ratio of 3 : 2 ; if it contains 3 gallons more wine than water, what is the quantity of wine in the mixture ?

83. 3 men and 6 boys can do 4 times as much work as a man and a boy can do, in the same time. Find the ratio of the works done by a man and a boy in the same time.

84. A mixture is composed of 4 parts brandy and 1 part water ; one gallon of water is added, and the mixture contains 3 times as much brandy as water : find the quantity of brandy in the mixture.

85. A mixture contains wine and water in the ratio of 3 : 2, another contains wine and water in the ratio of 4 : 5 ; how many gallons of the latter must be mixed with 3 gallons of the former that the resulting mixture may contain equal quantities of wine and water ?

86. A, B and C are three vessels holding 1, 2 and 4 gallons respectively. A is empty, B is full of water and C is full of wine. A is filled from B, B is replenished from C, and then A is emptied into C. When this operation has been performed once more, what will be the ratio of the wine in B to the water in C?

87. An alloy of silver is mixed with an alloy of gold in the ratio of ·73 to ·37; the quantity of dross in the silver alloy is 12 parts in 100, and in the gold alloy 15 parts in 100: compare the quantities of gold, silver and dross in the mixture.

88. A barters some sugar with B for flour which is worth 2*s*. 3*d*. per stone, but uses a false stone weight of $13\frac{1}{2}$ lb.; what value should B set upon his flour, that the exchange may be fair?

89. If the work done by a man, a woman, and a child be in the ratio of 3, 2, 1, and there be in a factory 24 men, 20 women and 16 children, whose weekly wages amount to R224, what will be the yearly wages of 27 men, 40 women and 15 children?

90. A lb. of tea and 3 lb. of sugar cost R3, but if sugar rose 50 per cent. and tea 10 per cent., they would cost R3. 8*a*.; find the prices per lb. of tea and sugar.

91. A bankrupt has goods worth R9750; and had they realised their full value, his creditors would have received 13 annas in the rupee; but $\frac{2}{5}$ths were sold at 17·5 p. c., and the remainder at 23·75 p. c., below this value. What sum did the goods fetch, and what dividend was paid?

92. Gold is sold at the Mint at £3. 17*s*. 9*d*. per oz., and is mixed with alloy, worth 5*s*. 2*d*. per oz., in the ratio of 11 : 1. If sovereigns be coined of this mixture, each weighing 5 dwt. 3·47 gr., what is the Mint profit per 100 sovereigns?

93. A bag contains 160 coins consisting of half-crowns, shillings, sixpences and fourpences, and the values of the sums of money represented by each denomination of coin are the same; how many of each are there?

94. In sending 100 cheroots to England I paid freight $\frac{2}{3}$ of their prime cost; landing charges $\frac{1}{5}$ of their cost including freight; and duty $2\frac{1}{2}$ times their cost including freight and landing charges. Altogether the cheroots, duty paid, in London cost me £7. What did I give for them in Madras?

95. A number of rupees is divided amongst four men. A receives $\frac{2}{3}$ of the whole, B $\frac{3}{4}$ of the remainder, C $\frac{4}{5}$ of what then remains, and the number of rupees given to D is the square root of the whole number to be divided. What sum does each receive?

96. For $\frac{2}{3}$ of the distance up a ghaut the rise is 1 foot in 24 (measured along the road) and for the remaining third the rise is 1 in 16. The top of the ghaut is 1,400 ft. above the bottom; what is its length?

97. In a company of 100 people, of whom some are rich and some are poor, the rich subscribe and give 1*a.* 3*p.* to each poor man; this costs the rich men 7*a.* 1*p.* each: how many rich and how many poor men are there?

98. Given that gold is worth £3. 17*s.* 10*d.* per oz., and silver 4*s.* 10*d.* per oz., and that the weights of equal volumes of gold and silver are as 19 : 11; find the volume of silver equal in value to a cubic inch of gold.

99. A tradesman bought a quantity of goods, and sold $\frac{2}{3}$ of them at a profit of 10 p. c.; the price rising, he got $12\frac{1}{2}$ p. c. profit on the remainder, and on the whole gained ₨425: what sum did he lay out?

100. A publican buys two butts of wine, one for ₨1200, and one for ₨1100; he also buys a third and after mixing the three, retails the wine at ₨22. 8*a.* a dozen, making $12\frac{1}{2}$ p. c. on his outlay: supposing the number of dozens in a butt to be 52, find the price of the third butt.

101. A merchant sells 49 quarters of wheat at a profit of 7 p. c., and a certain number of quarters at a profit of 11 p. c. The cost price of a quarter of wheat being £3. 12*s.* 6*d.*, he would have lost £2. 10*s.* 9*d.* if he had sold the whole at a profit of 9 p. c. Find the total number of quarters of wheat sold by him.

102. The shares in a banking concern are ₨1000 each, ₨426. $10\frac{2}{3}$*a.* are only paid up, and the shares are quoted in the market at ₨460. The dividend is ₨$7\frac{1}{2}$ per share quarterly. A gentleman holds 100 original shares. Find what interest he makes per cent.; and how much per cent. would he make, if he sold out and invested in 4 per cent. Govt. stock at par?

10 . A person finds that if he invest a certain sum in railway shares paying £6 per share when the £100 share is at £132, he will obtain £10. 16*s.* a year more for his money than if he invest in 3 per cent. consols at 93. What sum has he to invest?

104. A person has ₨24,180 to invest; the $5\frac{1}{2}$ per cent. Govt. loan being at 108 and the 6 per cent. Municipal loan of ₨1,000 being at 1020; find how he must divide his capital between the Govt. and Municipal loans, that he may obtain the same income from each.

105. A railway proprietor receives one year a dividend of 6 per cent. on his stock, and pays an income-tax of 4*d.* in the £. The next year he receives a dividend of $6\frac{1}{2}$ per cent. and pays an income-tax of 3*d.* in the £, and finds that his net income is £249 more. How much railway stock does he hold?

106. A man sold at 48 and 95 respectively £500 ordinary stock in the *A* Railway paying a dividend at the rate of $1\frac{3}{4}$ and £800

preference stock in the *B* Railway paying a dividend of 4 per cent. He then invested $\frac{1}{4}$ of the money in the Tramway Company where the £24 share paying interest at 6 per cent. was at £6 premium; £150 in the *C* Railway which paid no interest; and the remainder in Bank shares at par: what rate of interest must he receive from the Bank in order to increase his annual income by £12. 5*s.*?

107. There are two railway engines whose rates of motion may be represented by 1 and ·75. Supposing the slower to have been 12 miles in advance of the faster train on the same line, how far would the faster train have to travel before it overtook the other?

108. The value of 1 lb. of gold is 20 times that of 1 lb. of silver and the weights of equal volumes of gold and silver are as 19 : 10; find the value of a bar of silver equal in bulk to a bar of gold of value £380.

109. A merchant owes a bill of R5,795, payable in 8 months and another of R7,822, payable in 12 months; he takes up these two bills and gives in their place one for R13,716, payable in 12 months: what is the rate of interest per cent. per annum?

110. A Calcutta merchant has to pay R10,512. 8*a.* to his agent in Bombay. What must he give for a bank draft to that amount, exchange being at $100\frac{1}{4}$?

111. A man bequeaths his property amounting to R49,166 in such a way that $\frac{1}{3}$ of his wife's share, $\frac{2}{5}$ of his eldest son's, $\frac{3}{8}$ of his younger son's and $\frac{1}{2}$ of his daughter's share are all equal. Find the share of each.

112. *A* and *B* exchange goods; *A* gives 13 cwt. of hops, the retail price of which is 56*s.* per cwt. but in barter he rates them at £3. *B* gives 10 barrels of beer, the retail price of which is 1*s.* a gallon, but the value of which he raises in proportion to the increased price of the hops. How much must *B* give in money?

113. A person having to pay R10,572 two years hence, invests in the 4 per cent. Transfer Loan to accumulate interest till the debt shall be paid, and also an equal sum the next year.[1] Supposing the investment to be made when paper is at $86\frac{1}{2}$, and the price to remain the same, what sum must be invested on each occasion that these may be just sufficient to pay the debt at the given time?

114. A train has been travelling 20 miles an hour: the steam power is doubled, whilst from various causes the resistance of the train is increased by one-half. (The original steam power is three times the resistance). At what rate will the train now travel?

115. A sailing vessel reaches Madras from Calcutta in 6 days; a steamer whose speed is to that of the sailing vessel as 3 : 2 starts at the same time, but meets with detentions that average 6 hours daily. Which will reach Madras first? And by how much?

116. A book containing between 900 and 1000 pages is divided into four parts, each part being divided into chapters. The whole number of pages in each of the four parts is the same. Each chapter in the first part contains 20 pages, each chapter in the second 40, each chapter in the third 60, and each chapter in the fourth 80. Find the whole number of chapters in the book.

117. A person buys a piece of land at £25 an acre, and by selling it in allotments find that the value is increased by one-half, so that, after reserving 20 acres for himself, he clears £200 on his purchase money by the sale of the remainder. How many acres were there?

118. Find how much rice a family requires monthly, when a reduction in the price from 7 to 10 measures for the rupee reduces the total monthly expenses from ₨$31\frac{1}{2}$ to ₨30.

119. *A* barters sugar with *B*, for rice which is worth $1\frac{7}{8}$ annas a measure, but in weighing his sugar uses a false maund weight. *B* discovers this, and to make the exchange fair raises the price of his rice to $2\frac{1}{2}$ annas a measure. Find the real weight of the false maund which *A* uses.

120. A person pays an income-tax of 4*d.* in the £ during the first half of the year and of 3*d.* in the £ during the second half, and finds that owing to an increase in his income he pays the same amount of tax for the second as for the first half of the year. If his gross income for the year is £700, find his net income.

121. The materials of an old building were sold for ₨1,500 upon condition that they should be removed within 30 days under a penalty of ₨10 per day for every day beyond 30 days. The purchaser employed 40 men at $3\frac{1}{2}$ annas per day to do the work, and after selling the materials for ₨2365, he cleared ₨190 by his bargain. Find the number of days the men were at work.

122. *A* and *B* enter into partnership; *A* supplies the whole of the capital, amounting to ₨45,000 upon condition that the profits are to be equally divided, and that *B* pays *A* interest on half the capital at 10 per cent. per annum but receives ₨120 per mensem for carrying on the concern. Find their total yearly profits when *B*'s share is equal to $\frac{1}{2}$ of *A*'s share.

123. If the value of a rupee varies from 1*s.* 9*d.* to 1*s.* $9\frac{1}{2}d.$ and of the franc from $9\frac{1}{2}d.$ to 10*d.*; find the maximum number of francs which it is always safe to give for ₨500.

124. If the volume of a sphere $=\frac{4}{3}\times 3{\cdot}1416\times$ the cube of the radius, find how many spherical balls each $\frac{1}{3}$ inch in diameter can be made out of a cubic inch of clay, and how much clay will remain over.

125. Paper-money is at a discount of 10 per cent. A man buys goods marked £27 (paper-money) and offers that sum in gold. How

much paper-money must he receive in change, 10 per cent. abatement being allowed for cash ?

126. A reservoir is to be emptied, the rate of discharge of the contents being diminished by 100 gallons every hour. The first half will be emptied in 3 hours, the second in 4 hours. How many gallons does the reservoir contain ?

127. What must be the least number of soldiers in a regiment to admit of its being drawn up 2, 3, 4, 6 or 8 deep, and also of its being formed into a solid square ?

128. *A*, *B* and *C* are partners. *A* receives $\frac{2}{3}$ of the profits, *B* and *C* dividing the remainder equally. *A*'s income is increased by R400 when the rate of profit rises from 5 to 7 per cent. Find the capital of *B*.

129. How many years' purchase should be given for an estate so as to get 4 per cent. for the money ?

130. An agent has to receive a rent paid in corn from a tenant, and to deliver it to the landlord. At each payment he uses, so as to benefit himself, a false balance, such that 4 seers in one scale balance 5 seers in the other. Corn being worth R2. 8*a.* a md., the value of his plunder is R4. What is the corn-rent ?

131. A zemindary is bought at 20 years' purchase for R27000, one-third of the purchase-money remaining at mortgage at 9 per cent. The cost of collecting rents is R140 per annum. What interest does the purchaser make on his investment ?

132. A baker's outlay for flour is 70 per cent. of his gross receipts, and other trade expenses amount to $\frac{1}{5}$ of his receipts. The price of flour falls 50 per cent., and other trade expenses are thereby reduced 25 per cent. By how much should he now reduce the price of a 5*d.* loaf to make the same amount of profit ?

133. 1000 copies of a pice newspaper weigh $\frac{1}{2}$ of a maund, and when the paper duty was removed the profit on the receipts was increased 5 per cent. What was the duty per md. on paper ?

134. A horse was sold at a loss of 10 p. c. ; if it were sold for R70 more there would have been a gain of 4 per cent. : for how much was the horse sold ?

135. A contractor sends in a tender of R7000 for a certain work ; a second sends in a tender of R6950, but stipulates to be paid R3000 at the end of a month ; find the difference between the tenders, supposing the work to be finished in 3 months, and money to be worth $\frac{1}{2}$ per cent. per month simple interest.

136. A labourer was engaged for 20 days, on the agreement that for every day he worked he should have 4*a.*, but that for every day he absented himself he would be fined 1*a.* He received R2. 13*a.* at the end of the time : how many days was he absent ?

137. A man was hired to do a certain amount of work, on the condition that for every day he worked he should have 12*a*., but that for every day he absented himself he should lose 4*a*. He worked 3 times as many days as he absented himself, and received on the whole ₨10. How long was he doing the work?

138. A grocer buys two maunds of sugar; he sells one maund at a profit of 10 p. c., and the other which cost ₨2. 8*a*. more, at a profit of 15 p. c. If the retail price per seer of the latter be $1\frac{3}{10}a$. more than that of the former, find the cost price of each maund.

139. A shop-keeper buys 2 md. of sugar, and 1 md. more of a superior kind, giving ₨1. 8*a*. a md. more for the latter than the former. He retails it, when mixed, at 4 annas a seer, and makes a profit of 25 p. c. on his outlay. What did he give per md. for each kind of sugar?

140. Two boys begin to count two equal piles of rupees. One counts 5 while the other counts 4. When the former has just finished the latter has 6 left. What is the number of rupees in each pile?

141. The price of a yard of jean is $\frac{2}{3}$ of the price of $2\frac{1}{4}$ yd. of longcloth; and the weight of 5 yd. of jean is $\frac{5}{6}$ of the weight of 8 yd. of longcloth. If the price of 2 lb. of jean be ₨3, what is the price of $1\frac{1}{2}$ lb. of longcloth?

142. Three tramps meet together for a meal: the first has 5 loaves, the second 3, and the third, who has his share of the bread, pays the other two 8 half-pence; how ought they to divide the money?

143. *A* and *B* barter: *A* has 7 md. of flour worth ₨3. 8*a*. a md., but insists on having ₨3. 12*a*. a md.: *B* has rice worth ₨1. 5*a*. a measure, which he raises in price in proportion to *A*'s demand. *A* receives 16 measures of rice: what cash does he get besides?

144. *A* and *B* barter: *A* has 200 lb. of tea worth 2*s*. 6*d*. a lb. but insists on 2*s*. 9*d*. a lb.: *B* has coffee worth 1*s*. 9*d*. a lb.: how much must he raise the price so that *A* gets £5. 2*s*. and 2 cwt. of coffee?

145. A river 14 ft. deep, 182 yd. wide flows at the rate of 3 miles an hour; (i) how many tons, (ii) how many gallons of water, pass a certain point per minute? [A cu. ft. of water weighs $62\frac{1}{2}$ lb.; a gallon contains $277\frac{1}{4}$ cu. in.]

146. A four-wheeled carriage travels round on a circular railway. The circumferences of the two wheels of the carriage and of the two circles of rails are proportional to 6, 7, 7000, 7014. Find the number of revolutions made by each of the four wheels in a complete circuit.

147. Eleven boys fired 10 shots each at a target, and scored

286; 20 bull's-eyes were made and 11 misses; how many centres and outers were there? (A bull's-eye scores 4, a centre 3, an outer 2).

148. The sum of £177 is to be divided among 15 men, 20 women and 30 children, in such a manner that a man and a child may receive together as much as two women, and all the women may together receive £60; what will they each respectively receive?

149. *A* owed *B* three-fourths of what *B* owed *C*; to settle matters, *B* gave ₨2 to *A* who then paid *C*; what did *B* owe *C*?

150. A man for 4 years spends ₨500 a year more than his income. At the end of that time, he reduces his expenditure 30 per cent. and in 3 years pays off his debt and saves ₨1000. What is his income?

151. A tree grows 2 yards in its first year, and afterwards it grows each year 1 foot less than it did the previous year. The value of the tree at any time is equal to the number of rupees in the square of the number of yards in its height; find the value of the tree when it has done growing.

152. If standard gold, worth £3. 17*s*. $10\frac{1}{2}$*d*. per ounce be so far alloyed as to be worth only £3. 16*s*. $1\frac{1}{2}$*d*. per ounce, find the least integral number of sovereigns made of the alloyed gold, which shall be equal in value to an exact number made of the standard gold.

153. Find the least integral number of ounces of pure silver, worth ₨2. 14*a*. $6\frac{6}{11}$*p*. per ounce, that, with the proper proportion of alloy, can be coined into an exact number of rupees.

154. Mahogany is 50 lb. to the cubic foot, water is $62\frac{1}{2}$ lb., and iron is $7\frac{1}{2}$ times as heavy as water; what thickness of iron will weigh as much as a 6-inch plank of mahogany?

155. A sum of ₨62 is to be divided among 10 men, 15 women, 8 boys and 12 girls. For every rupee that a man gets, a boy gets 6 annas, and for every half-rupee that a woman gets, a girl gets 2 annas. The whole money obtained by the boys is equal to that obtained by the girls. How much does each person get?

156. A wooden closed box, made of $\frac{1}{2}$-inch plank, is externally 15 in. long, 10 in. broad and 6 in. high. The box weighs 6 lb. when empty, and 86 lb. when filled with mercury. Compare the weights of equal bulks of the wood and mercury.

157. ₨430 is divided among 45 persons consisting of men, women and children. The sums of the men's, women's and children's shares are as 12 : 15 : 16, but the individual shares of a man, woman and child are as 6 : 5 : 4. Find the number of men, women and children.

158. Bronze contains 91 per cent. of copper, 6 of zinc, and 3

of tin. A mass of bell-metal (consisting of copper and tin only) and bronze fused together is found to contain 88 per cent. of copper, 4·875 of zinc, and 7·125 of tin. Find the proportion of copper and tin in bell-metal.

159. An alloy contains 12 parts by weight of lead, 4 of antimony, and 1 of tin. How much of this alloy must be taken, and how much lead and tin added to it to make up 9 cwt. of type-metal consisting of 14 parts lead, 3 antimony and 1 tin?

160. Three persons *A*, *B*, *C*, finished a piece of work. *A* worked at it for 5 days, *B* for 7 days and *C* for 9 days. Their daily wages were as 4 : 3 : 2, and the total earnings amounted to ₨7. 6*a*. What were the daily wages of each?

161. Two passengers are charged for excess of luggage ₨1. 8*a.* and ₨5. 4*a.* respectively. Had the luggage all belonged to one person he would have been charged ₨7. 8*a.* for excess. How much is allowed free, the charge for excess being 12*a.* per md.?

162. If the cost of making bread be one rupee per bushel of wheat, what is the price of wheat when the two-anna loaf is twice as large as it is when wheat is ₨5 a bushel?

163. If the rate of wages vary as the price of rice, and if 57 men working for 35 days receive ₨405. 3*a.* 9*p.* when rice is sold at the rate of 136 measures for ₨39; find the price of rice per measure when 70 men working for 19 days receive ₨353. 4*a.* 6*p.*

164. There is a leak in the bottom of a cistern. When the cistern was in thorough repair, it would be filled in $2\frac{1}{2}$ hours. It now takes half an hour longer. If the cistern is full, how long would it be in leaking itself empty?

165. *A* can do $\frac{2}{3}$ of a piece of work in $\frac{3}{4}$ of the time in which *B* can do $\frac{4}{5}$ of it, and *B* can do $\frac{3}{4}$ of it in $\frac{2}{3}$ of the time that it would take *C* to do another piece of work one-fourth as large again as the first. If *C* can finish the former piece of work in 10 hours, how long would it take *A* and *B* together to do it?

166. *A* and *B* start on a journey at the same time. *B* travels at $\frac{4}{7}$ of *A*'s rate, and arrives 3 hr. 15 min. after him. In what time did each complete the whole journey?

167. The expenses of a family when rice is at 20 seers for a rupee are ₨50 a month; when rice is at 25 seers for a rupee the expenses are ₨48 a month; what will they be when rice is at 30 seers for a rupee?

168. A man who can walk down a ghaut at the rate of $4\frac{1}{2}$ and up it at the rate of $3\frac{1}{4}$ miles an hour, descends and returns to his starting point after walking for 2 hours 4 minutes. How far did he walk?

169. An express train owing to a defect in the engine goes at

$\frac{5}{8}$ of its proper speed, and arrives at 6-49 P. M. instead of 5-55 P. M.; at what hour did it start?

170. A person going from Pondichery to Ootacamond travels 90 miles by steamer, 330 miles by rail and 30 miles by horse-transit. The journey occupies 30 hr. 50 min., and the rate of the train is 3 times that of the horse-transit and $1\frac{1}{2}$ times that of the steamer. Find the rate of the train.

171. A person walks from A to B at the rate of 3 miles an hour and after transacting some business which occupies him an hour, returns to A by the tramway at the rate of 5 miles an hour. He then finds he has been absent 2 hours 20 minutes. Find the distance from A to B.

172. The expenses of a family, when rice is 12 seers for a rupee, are ₨50 a month; when rice is 14 seers for a rupee, the expenses are ₨48 a month (other expenses remaining unaltered): what will they be when rice is at 16 seers per rupee?

173. A bankrupt has book-debts equal in amount to his liabilities, but on ₨8640 of such debts he can recover only $8\frac{1}{2}$ annas in the rupee, and on ₨6300 only $5\frac{1}{4}$ annas in the rupee. After allowing ₨1054 . 11 . 0 for the expenses of bankruptcy, he finds that he can pay his creditors 12 annas in the rupee. Find the total amount of his debts.

174. A train starts with a certain number of passengers. At the first station it drops $\frac{1}{3}$ of these and takes in 20 more. At the next it drops $\frac{1}{2}$ of the new total and takes 10 more. On reaching the third station there are 60 left. What number started?

175. One pound troy of standard silver which contains 37 parts in 40 of fine silver is coined into 66 shillings. If the value of pure silver rises 10 per cent., what must be the reduction of pure silver in a shilling?

176. A landlord has an estate worth ₨40000 a year, but has to pay $\frac{1}{2}$ anna in the rupee on the gross income for taxes. He sells it at 20 years' purchase on the gross income, and invests the proceeds in the 4 per cents. at 95. What is the difference in his income?

177. In firing at a mark A hits in 2 out of 4 shots, B in 3 out of 5, and C in 4 out of 7. The mark was hit 468 times. Supposing each to have fired the same number of shots, find how many hits each made and the total number of shots fired.

178. A shop-keeper buys sugar at ₨12. 8*a*. a md.; at what price must he sell it to gain 8 per cent., and allow a purchaser 10 per cent. discount?

179. In a manufactory 100 coolies work for 4 days a week, but on the remaining 3 days some are absent; the weekly wages of

the coolies are thus reduced in the ratio of 32 : 35. Find the number of absentees.

180. The manager of a boarding house having already 50 boarders, finds that an addition of 10 increases the gross monthly expenditure by ₨20, but diminishes the average cost per head by ₨1. What did the monthly expenses originally amount to?

181. If 9 oz. of gold, 10 carats fine, and 5 oz., 11 carats fine, be mixed with 6 oz. of unknown fineness, and the fineness of the resulting mixture be 12 carats, what was the unknown fineness?

182. A tradesman's stock in trade is valued on January 1st, 1868, at £8,000, he has also £350 in cash and owes £1,870; during the year his personal expenses, £300, are paid out of the proceeds of his business, and on January 1st, 1869, his stock is valued at £7,950, he has £570 in cash and owes £1,510. What is the whole profit on the year's transactions after deducting 5 per cent. interest on the capital with which he began the year?

183. If 20 English navvies, each earning 3*s.* 6*d.* a day, can do the same piece of work in 15 days that it takes 28 foreign workmen, each earning 3 francs a day, to complete in 20 days; taking the value of the franc at 10*d.*, determine which class of workmen it is most profitable to employ. If a piece of work done by the navvies cost £3,000, what would be the cost of the same work done by foreign workmen?

184. A merchant in New York wishes to remit to London 5110 dollars, a dollar being equal to 4*s.* 6*d.* English: for what sum in English money must he draw his bill when bills *on* London are at a premium of $9\frac{1}{2}$ per cent.?

185. A person borrows £100, and at the end of each year pays £25 to reduce the principal and to pay interest at 4 per cent. on the sum which has been standing against him through that year. How much will remain of the debt at the end of 3 years?

186. If a metric system of area were adopted wherein 1 acre 1 rood 3 perches is represented by 5·12, express the unit of measurement in sq. yards and decimal parts of a sq. yd.

187. If gold weighs 19 times as much as water, and silver 12 times as much, find how many times heavier than water is a coin which contains 10 parts of gold and 1 of silver.

188. A certain reef of quartz when crushed yields ·0011 per cent. of gold. If the working expenses amount to 62·5 per cent. of the gross receipts, and the net profit on each 100 tons is £52. 10*s.*; find the number of grains in a sovereign.

189. A certain article of consumption is subject to a duty of 6*s.* per cwt.; in consequence of a reduction in the duty the consumption increases one-half, but the revenue falls one-third. Find the duty per cwt. after the reduction.

190. If the duty on a certain commodity were reduced 25 per cent., by how much per cent. must the consumption be increased that the same revenue may be derived from it?

191. If 2 cu. in. of gold together with 3 cu. in. of silver are equal in weight to 74 cu. in. of water, and the weights of equal volumes of gold and water be represented by the numbers 19 and 1, what number represents the weight of an equal volume of silver?

192. A farmer bought equal numbers of two kinds of sheep, one at £3 each, the other at £4 each. If he had expended his money equally in the two kinds he would have had 2 sheep more than he did; find how many he bought.

193. A man travels 150 miles in 13 hours, partly by rail and partly by steamer; if he had gone all the way by rail, he would have ended his journey 8 hours sooner, and saved $\frac{4}{5}$ of the time he was on steamer; how far did he go by rail?

194. In a distilling operation, during 3 hours the fluid contained 70 per cent. of alcohol, during $2\frac{1}{2}$ hours 60 per cent., and during the remaining $1\frac{1}{2}$ hours 40 per cent. What is the average strength of the whole fluid distilled over, assuming that it came over at a uniform rate during the whole time?

195. During a distillation the fluid that comes over in 3 consecutive hours contains 47,35 and 20 per cent. of alcohol respectively. The rates at which it comes over during these 3 hours are in the ratios of 2, 3 and 4. What is the percentage of alcohol in the whole mixture?

196. I bought a number of mangoes at 35 for R2. I divided the whole into two equal parts, one of which I sold at 17, and the other at 18 mangoes per R1. I spent and received an integral number of rupees, but bought the least possible number of mangoes. How many did I buy?

197. Find the cost in rupees of one mile of railway, which consists of two rails, each weighing 40 lb. per yard, on wooden sleepers, weighing 70 lb. each, placed 2 ft. 8 in. apart. The rails cost in England £6 . 13 . 0 per ton and the sleepers 2*s*. $4\frac{1}{2}$*d*. each. The rate of freight is £1 . 5 . 0 per ton, and landing charges amount to R2. 8*a*. per ton. Rate of exchange 1*s*. 8*d*. per rupee.

198. The length of the E. B. Railway being 110 miles and the capital employed in its construction 1500000*l*., what must be the gross annual trafic receipts per mile in order that a dividend of 5 per cent. may be paid to the share-holders after allowing 45 per cent. of the gross receipts for current expenditure?

199. A person in India sells a bill on London for 358*l*., payable at 3 months' sight at the rate of 1*s*. $10\frac{3}{4}$*d*. per rupee. The purchaser requires payment on presentation; what amount does he receive after discount at 5 per cent. has been deducted?

200. The Guernsey pound contains 18 oz. avoir., and the Guernsey shilling contains 13 English pence. If a Guernsey pound of butter cost 1*s.* 6*d.*, Guernsey money, what will be the price in English money of $2\frac{1}{2}$ lb. avoir.?

201. A contractor employs a fixed number of men to complete a work. He may employ either of two kinds of workmen : the first at 26*s.* 6*d.* per week each, the second at 18*s.* 6*d.* per week each : the work of the one of the former being to that of one of the latter as 5 to 4. If he finishes it as quickly as possible, he spends £270 more than he would have done if he had finished it as cheaply as possible, but takes 4 weeks less time. What would it have cost if he had employed equal numbers of the two kinds of workmen?

202. A manufactory turns out 50 tons of iron goods weekly, using up for that purpose 51 tons of iron at £6. 15*s.* per ton, 100 tons of coal at 11*s.* 6*d.* per ton, and £45 worth of other materials ; rent, rates and taxes amount to £219 annually ; wages and incidental expenses to £75 per week. At what price per cwt. must the iron be sold in order that the works may gain 8 per cent. per annum on a capital of £35000? [Reckon 52 weeks to the year.]

203. Two lumps, composed of gold, silver and copper, together weigh 10 oz. ; one lump contains gold 75 p. c. and silver 15 grains per oz., the other contains gold 85 p. c. and silver 12 grains per oz. The total quantity of silver in the two lumps is 141 grains. If the two lumps are melted and formed into one, what per cent. of gold will it contain?

204. The only three creditors of an insolvent whose assets amount to £100 and who can pay only 5*d.* in the £, agree among themselves to take dividends in the proportion of the number of £. *s.* and *d.* respectively, contained in the amounts due to them. The dividends thus taken are in the proportion of 12 : 7 : 6. What are the amounts of their debts?

205. At an examination $\frac{1}{5}$ of a class gains $\frac{7}{8}$ of the maximum number of marks, $\frac{1}{10}$ gains $\frac{3}{4}$, $\frac{2}{5}$ gains $\frac{1}{2}$, $\frac{1}{4}$ gains $\frac{1}{4}$, and the rest $\frac{1}{8}$. The average number of marks gained by the whole class is 166 ; what is the maximum?

206. A mass of gold and silver weighing 9 lb. is worth £318. 13*s.* 6*d.* ; if the proportions of gold and silver in it were interchanged, it would be worth £129. 10*s.* 6*d.* ; it is known that 1 oz. of gold and 2 oz. of silver are worth £4. 8*s.* $1\frac{1}{2}$*d.* ; what is the price of gold and silver per oz.?

207. A person shooting at a target, distant 550 yards, hears the bullet strike the target 4 seconds after he fires. A spectator, equally distant from the target and the shooter, hears the shot strike the target $2\frac{1}{2}$ seconds after he heard the report ; find the velocity of sound.

208. A boatman rows 5 mi. with the tide in the time he would take to row 3 mi. against it; but if the hourly velocity of the current were $\frac{1}{2}$ a mile, he would row twice as rapidly with the tide as against it. Find his power of rowing in still water, and the velocity of the current.

209. A messenger sets out at the rate of 30 miles a day, but falls off in his speed 4 miles daily. Four days afterwards another sets off from the same place on the same route, travelling 50 miles the first day but falling off like the first 4 miles daily. After what time will one overtake the other?

210. Six months ago *A* invested £7620 in the 3 per cents. at $95\frac{1}{4}$, and six months hence he will receive £4300 four per cents. at 127. What is the present value of his property?

211. Two boats, *A* and *B*, row a race. *A* takes 4 strokes to *B*'s 5, but 6 of *B*'s are equal to 5 of *A*'s. *A* starts in front of *B* at such a distance that *B* must take 10 strokes to row over it. How many strokes must *B* take before overtaking *A*?

212. *A*, *B* and *C* run a mile race. *A* beats *C* by $76\frac{1}{2}\frac{3}{3}$ yards; *B* beats *C* by 11 seconds; the pace of *A* is to that of *B* as 45 : 44. In what time does each run the mile?

213. Three boys begin to fill a cistern; one brings a seer every minute, another 2 seers every 2 minutes, and the third 3 seers every 3 minutes. If the cistern holds 40 seers, in what time will it be filled?

214. *A* sells his goods 10 per cent. cheaper than *B*, and 10 per cent. dearer than *C*; how much would a customer of *B* save by taking R100 worth of goods from *C*?

215. Cannons are fired at intervals of 10 minutes in a town towards which a passenger train is approaching at the rate of 35 miles an hour; if sound travels 1142 feet per second, find at what intervals the reports will be heard by the passengers.

216. A man bought a horse and a carriage for R500, and sold the horse at a gain of 20 p. c. and the carriage at a loss of 10 p. c., thus gaining 2 p. c. on his whole outlay; for how much was the horse bought?

217. If 3 men and 5 women do a piece of work in 8 days, which 2 men and 6 children, or 5 women and 3 children, can do in 12 days, find the relative strength of men, women and children.

218. Three round balls revolve with equal velocities in three concentric circular grooves. They start from a position in which they are all in the same radius of the outermost circle. The innermost ball occupies 10 seconds in traversing its groove once. After what time will they all be again on a radius of the outermost circle, the radii of the grooves being proportional to the numbers 1, 3, 5?

219. Two guns are fired at the same place after an interval of 21 minutes, but a person approaching the place observes that 20 min. 14 sec. elapse between the reports; what was his rate of progress, sound travelling 1125 feet per second?

220. Ash saplings after 5 years' growth are worth 1*s*. 3*d*., and increase in value 1*s*. 3*d*. each year afterwards. For their growth they require each twice as many square yards as the number of years they are intended to grow before cutting. A plantation is arranged so that each year the same number may be ready for cutting. Find the greatest annual income which can be obtained per acre, allowing 20 per cent. for expenses.